The Spectral Method (MAPH 40260)

Part 2: The Advection Equation

Peter Lynch

School of Mathematical Sciences

Outline

Advection Equation

Finite Difference Approximation

Spectral Approximation

Solution of Advection Equation

Solution of Burgers' Equation

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The Advection Equation

We consider the simple advection equation in one dimension:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \,.$$

We will retain the continuous representation in time.

We will compare the grid point and spectral representation in space.

The contrast in the results is of great practical importance.

Outline

Grid Point Approximation

We evaluate the solution on a finite difference grid

$$u(m\Delta x, t) = U_m(t)$$

The equation becomes

$$\frac{\partial U_m}{\partial t} + c\left(\frac{U_{m+1} - U_{m-1}}{2\Delta x}\right) = 0\,.$$

We look for a solution of the form

$$U_m(t) = \exp[ik(m\Delta x - Ct)]$$

Substituting this into the equation, we have

$$-ikCU_m + \frac{ic}{\Delta x}\left(\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2i}\right)U_m = 0$$

That is,

$$-ikCU_m + \frac{ic}{\Delta x}(\sin k\Delta x)\,U_m = 0$$

This immediately leads to the result

$$C = \left(\frac{\sin k\Delta x}{k\Delta x}\right)c\,.$$

Clearly

$$C < c \qquad \text{for} \qquad k > 0\,.$$

For long waves, λ is large and k is small, so

$$C \approx c$$

For the shortest wave, $\lambda = 2\Delta x$ and $k\Delta x = \pi$, so

$$C = \left(\frac{\sin \pi}{\pi}\right)c = 0\,,$$

so the shortest wave is stationary.

For the $4\Delta x$-wave, $k\Delta x = \pi/2$, so

$$C = \left(\frac{\sin \pi/2}{\pi/2}\right)c = \left(\frac{2}{\pi}\right)c \approx \frac{2}{3}c\,,$$

so the wave is slowed down by about one third.

Outline

Spectral Approximation

Now consider the spectral approximation to

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \, .$$

We look for a solution

$$U(t) = \sum_k U_k(x,t) = \sum_k \exp[ik(x - Ct)]$$

Since the equation is linear, we can consider the individual components separately.

Substituting the solution in the equation, we get

$$-ikCU_k + ikcU_k = 0 \qquad \text{or} \qquad C = c \, .$$

The phase speed is represented exactly.

Outline

Solution of Linear Advection Equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \, .$$

Expand the solution in spectral components:

$$u(x,t) = \sum_{n=-N}^{+N} U_n(t) \exp(2\pi i n x/\ell) \, .$$

Note that we must truncate the expansion.

The truncation level N determines accuracy, just as the grid interval Δx does for the finite difference method.

Substituting in the expansion, the equation becomes . . .

$$\sum_{n=-N}^{+N} \left[\frac{dU_n}{dt} + \frac{2\pi icn}{\ell} U_n \right] \exp(2\pi inx/\ell) = 0 \,.$$

Now recall the orthogonality relationship

$$\frac{1}{\ell}\int_0^\ell \exp(-2\pi imx/\ell) \cdot \exp(+2\pi inx/\ell)\, dx = \delta_{mn} \,.$$

Multiply the equation by $\exp(-2\pi imx/\ell)$ **and integrate:**

$$\sum_{n=-N}^{+N} \left[\frac{dU_n}{dt} + \frac{2\pi icn}{\ell} U_n \right] \ell\delta_{mn} = 0 \,, \qquad \textbf{or}$$

$$\frac{dU_m}{dt} + \frac{2\pi icm}{\ell} U_m = 0 \,, \qquad m = -N, -(N-1) \ldots N-1, N \,.$$

The PDE has been reduced to a set of (independent) ODEs, which can easily be integrated.

Outline

Solution of Burgers' Equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \,.$$

This is the nonlinear advection equation with diffusion added to regularize the solution.

Expand the solution in spectral components:

$$u(x,t) = \sum_{n=-N}^{+N} U_n(t) \exp(2\pi inx/\ell) \,.$$

Substituting into the equation, expanding all terms and evaluating spatial derivatives analytically . . .

$$\sum_{n=-N}^{+N} \frac{dU_n}{dt} e^{\frac{2\pi inx}{\ell}} + \sum_{p=-N}^{+N} \sum_{q=-N}^{+N} U_p \left(\frac{2\pi iq}{\ell}\right) U_q \cdot e^{\frac{2\pi ipx}{\ell}} e^{\frac{2\pi iqx}{\ell}}$$
$$= \nu \sum_{n=-N}^{+N} \left(\frac{2\pi in}{\ell}\right)^2 U_n e^{\frac{2\pi inx}{\ell}} \,.$$

For simplicity, let us take $\ell = 2\pi$. **Then**

$$\sum_{n=-N}^{+N} \frac{dU_n}{dt} e^{inx} + \sum_{p=-N}^{+N} \sum_{q=-N}^{+N} iqU_pU_q e^{i(p+q)x} = -\nu \sum_{n=-N}^{+N} n^2 U_n e^{inx} \,.$$

We multiply by $\exp(-imx)$ **and integrate. The first and last sums reduce to single terms. The double sum reduces to a single sum.**

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{n=-N}^{+N}\frac{dU_n}{dt}e^{inx}\right)e^{-imx}\,dx=\sum_{n=-N}^{+N}\frac{dU_n}{dt}\delta_{mn}=\frac{dU_m}{dt}\,.$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left(-\nu\sum_{n=-N}^{+N}n^2U_ne^{inx}\right)e^{-imx}\,dx=-\nu m^2U_m$$

$$\frac{1}{2\pi}\int_0^{2\pi}\sum_{p=-N}^{+N}\sum_{q=-N}^{+N}iqU_pU_qe^{i(p+q)x}e^{-imx}\,dx=\sum_{p=-N}^{+N}i(m-p)U_pU_{m-p}$$

Lemma:

$$\sum_{p=-N}^{+N}i(m-p)U_pU_{m-p}=\tfrac{1}{2}im\sum_{p=-N}^{+N}U_pU_{m-p}$$

Proof:

$$\begin{aligned}\sum_{p=-N}^{+N}(m-p)U_pU_{m-p} &= m\sum_{p=-N}^{+N}U_pU_{m-p}-\sum_{p=-N}^{+N}pU_pU_{m-p}\\ &= m\sum_{p=-N}^{+N}U_pU_{m-p}-\sum_{q=-N}^{+N}qU_qU_{m-q}\\ &= m\sum_{p=-N}^{+N}U_pU_{m-p}-\sum_{q=-N}^{+N}(m-p)U_{m-p}U_p\end{aligned}$$

Taking the last term to the left, the lemma follows.

Burgers' Equation may now be written

$$\frac{dU_m}{dt}+\tfrac{1}{2}im\sum_{p=-N}^{+N}U_pU_{m-p}=-\nu m^2U_m\,.$$

Ignoring the nonlinear terms, we have

$$\frac{dU_m}{dt}=-\nu m^2U_m\,.$$

This means that each term gradually decays. The larger the wavenumber (the smaller the scale) the faster the decay rate. Viscosity acts most strongly on the smallest scales.

If we omit viscosity, we get the inviscid Burgers Equation:

$$\frac{dU_m}{dt}+\tfrac{1}{2}im\sum_{p=-N}^{+N}U_pU_{m-p}=0\,.$$

Again, Burgers' Equation in spectral form is:

$$\frac{dU_m}{dt}+\tfrac{1}{2}im\sum_{p=-N}^{+N}U_pU_{m-p}=-\nu m^2U_m\,.$$

We see that components interact in groups of three, called triads:

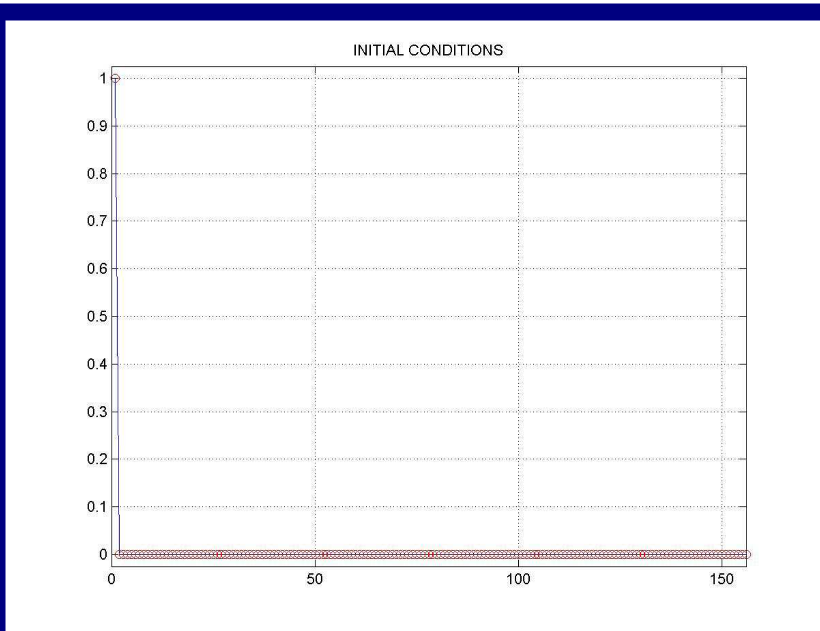
$$\left\{U_m \qquad U_p \qquad U_{m-p}\right\}$$

We see that all scales interact. For any mode U_m, any other mode U_p can change it by interacting with U_{m-p}. Energy can move from any scale to any other scale.

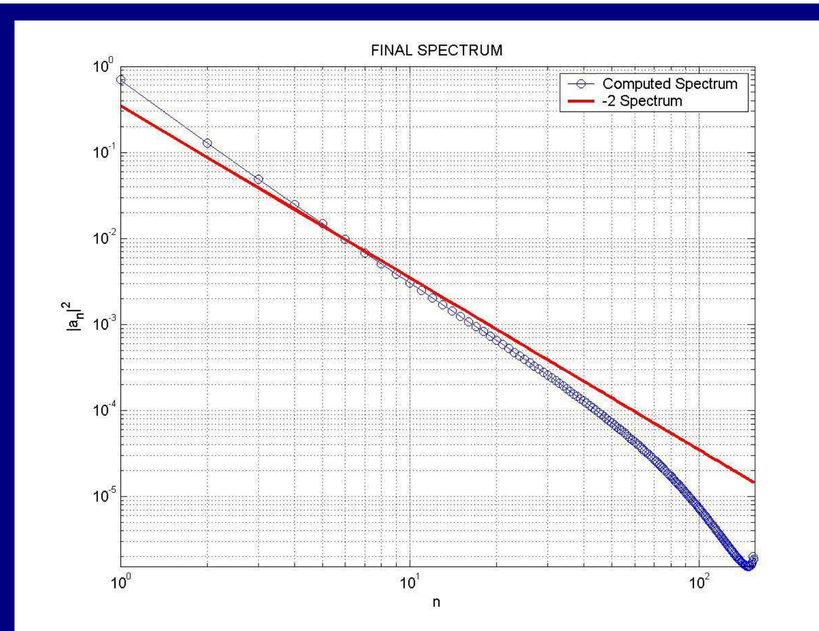
We may start with all the energy in the largest scale:

$$u(x,0)=U_1\left(\frac{e^{ix}-e^{-ix}}{2i}\right)=U_1\sin x\,,$$

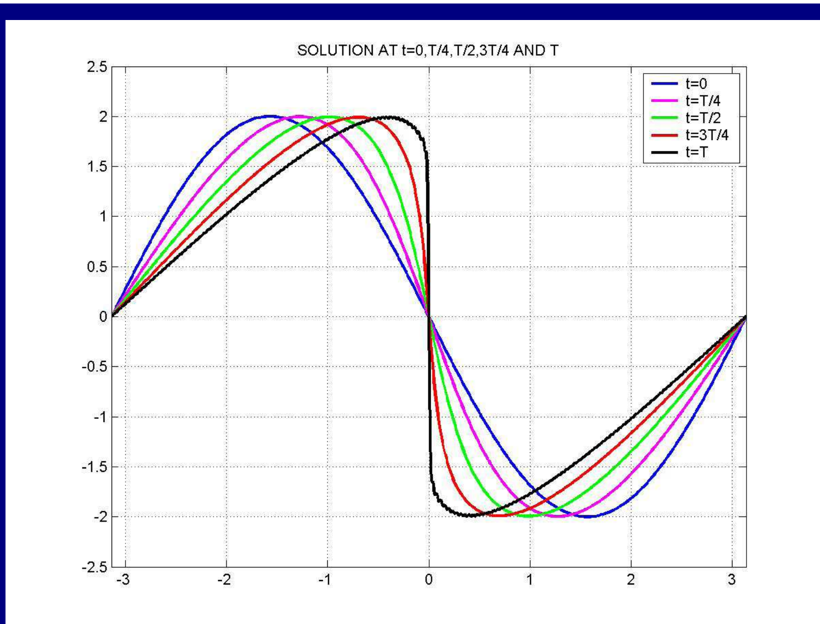
and the energy will quickly spread to other modes.



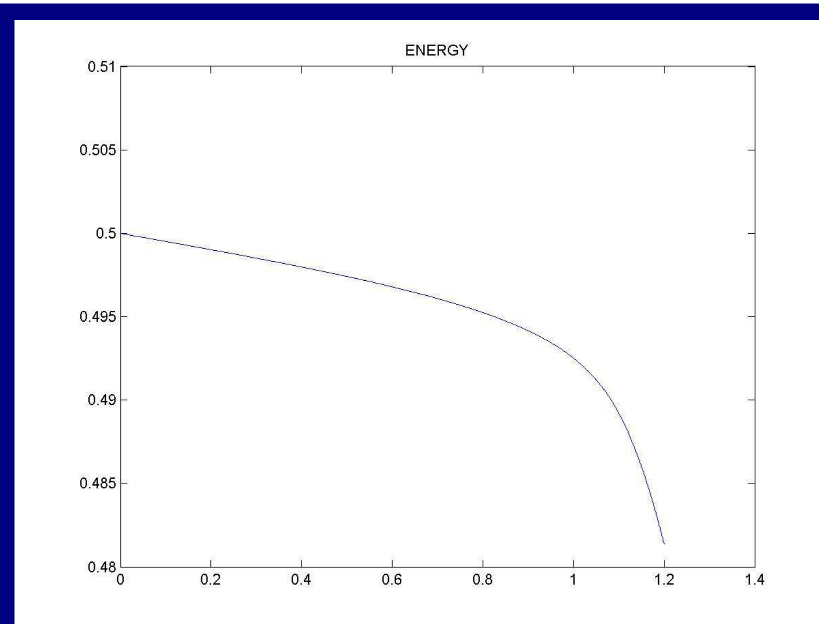
Initial conditions for Burgers' Equation.
Initial state is a pure sine-wave.



Final spectrum for Burgers' Equation.
Energy has spread to all modes.



Solution of Burgers' Equation. Shock has developed.
Initial state is a pure sine-wave.



Evolution of energy in time. Dissipation increases when energy reaches small scales.

End of Part 2