The Spectral Method (MAPH 40260)

Part 1: Spectral Analysis

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School of Mathematical Sciences

Outline

Introduction

Fourier Analysis

Vibrating String

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The Gridpoint Method

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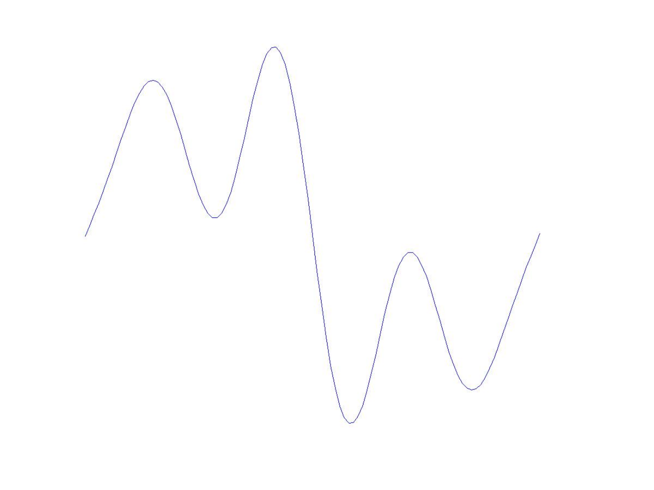
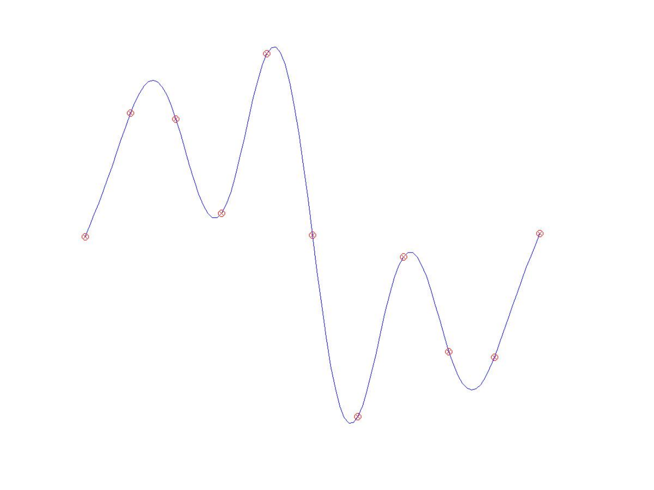
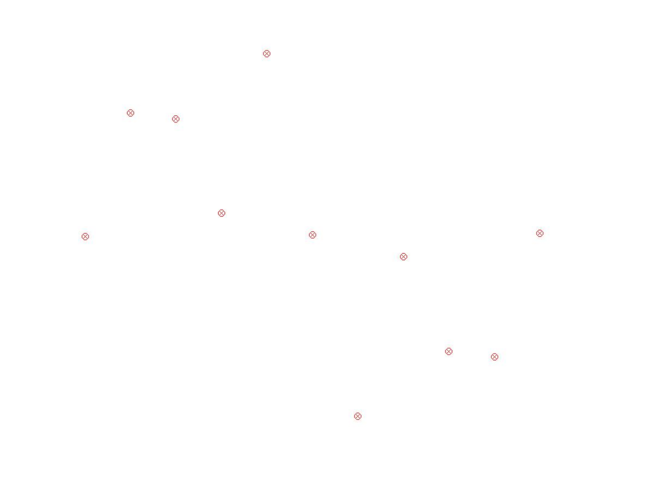
There are several answers to this question.



Continuous function of position.



Evalution on a set of grid points.



Grid point values.

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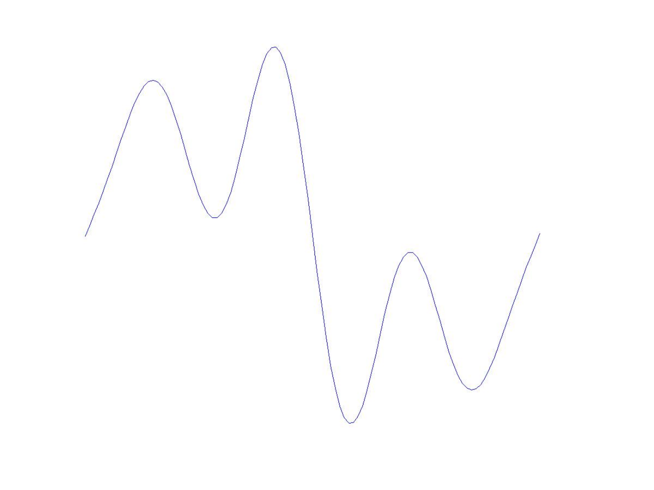
It is somewhat like splitting sunlight into the various colours of the spectrum.



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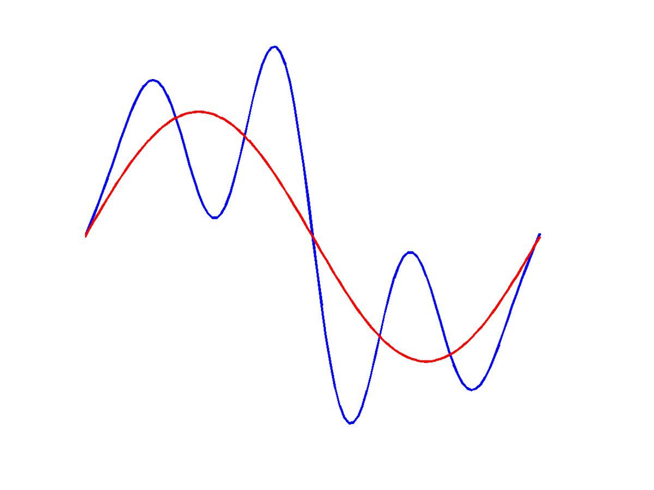
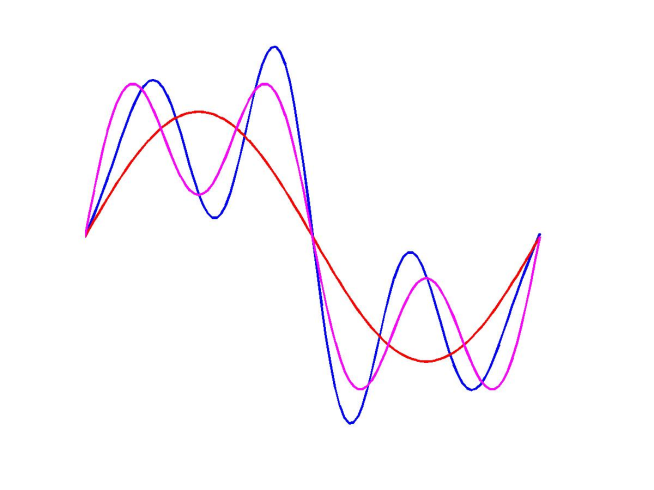
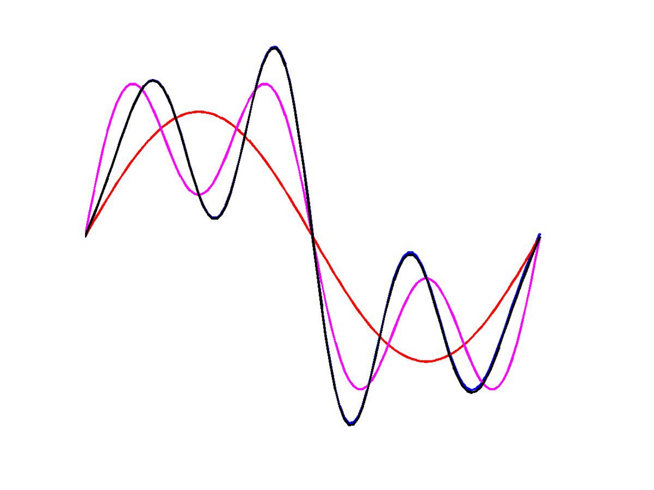


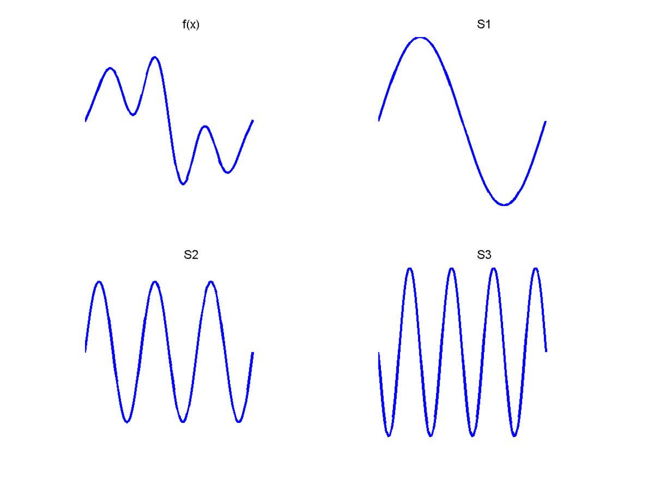
The first spectral component.



The first and second spectral components.



The first, second and third spectral components.



The original function and its three components.

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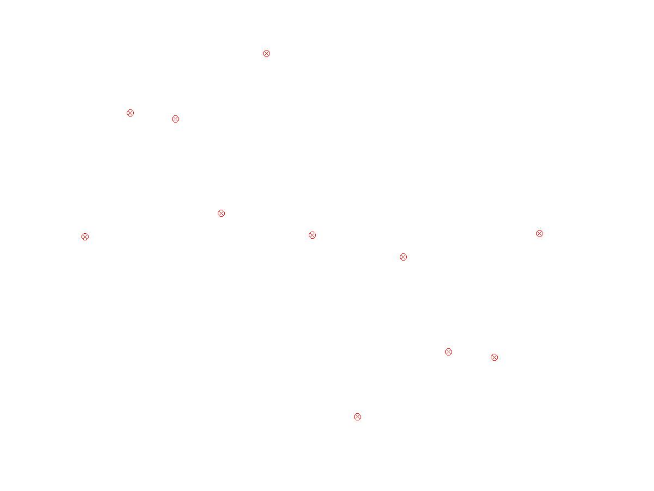
Big drawback: Evaluation of derivatives involves errors.



Grid point values. We have to get derivatives from this set of values.

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Big advantage: Evaluation of derivatives is exact.

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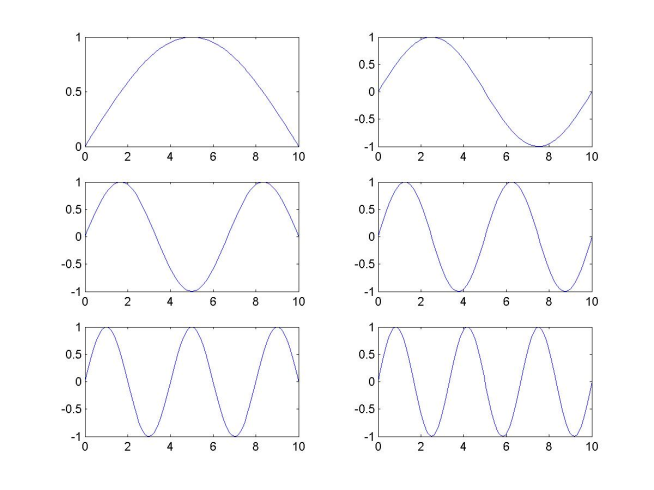
We note that sinusoidal functions with certain wavelengths also vanish at $x = 0$ and at $x = \ell$:

$$\sin \pi x/\ell \, , \sin 2\pi x/\ell \, , \ldots \, , \sin n\pi x/\ell \, ,$$

for all integer values of n.



The first six harmonic components ($\ell = 10$).

Orthogonality

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We easily show that

$$\begin{aligned}
\int_0^\ell [\Psi_n(x)]^2 \, dx &= \int_0^\ell \sin^2\left(\frac{n\pi}{\ell}x\right) \, dx \\
&= \frac{1}{2}\int_0^\ell \left[1 - \cos\left(\frac{2n\pi}{\ell}x\right)\right] \, dx \\
&= \left[\frac{x}{2}\right]_0^\ell - \left[\frac{\ell}{4\pi n}\sin\left(\frac{2n\pi}{\ell}x\right)\right]_0^\ell \\
&= \frac{\ell}{2} \, .
\end{aligned}$$

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\begin{aligned}
& \int_0^\ell \Psi_n(x) \cdot \Psi_m(x)\, dx \\
= \; & \int_0^\ell \sin\left(\frac{n\pi}{\ell} x\right) \cdot \sin\left(\frac{m\pi}{\ell} x\right)\, dx \\
= \; & \frac{1}{2} \int_0^\ell \left[\cos\left(\frac{n-m}{\ell}\pi x\right) - \cos\left(\frac{n+m}{\ell}\pi x\right)\right] dx \\
= \; & \frac{1}{2\pi} \left[\frac{\ell}{n-m} \sin\left(\frac{n-m}{\ell}\pi x\right) - \frac{\ell}{n+m} \sin\left(\frac{n+m}{\ell}\pi x\right)\right]_0^\ell \\
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We obtain an orthonormal set of functions:

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We might also choose $\Phi(x, t) = \Psi(x) \cos(\omega t)$ or $\Phi(x, t) = \Psi(x) \sin(\omega t)$.

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Hence

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k is the inverse of the wavelength, with a 2π factor.

The function

$$\Psi(x) = A \sin kx$$

is a solution of the o.d.e., and satisfies the boundary conditions if

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$\Psi_n(x)$ is an eigenfunction of the o.d.e. with eigenvalue $k_n = n\pi/\ell$.

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Thus,

$$A_m = \frac{2}{\ell}\int_0^\ell \Psi_m(x)\Psi(x)\,dx$$

Duality of the Fourier Transform

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There is a duality between $\Psi(x)$, a function in physical space and $\{A_n\}$, the coefficients in wavenumber space. Given either representation, we can obtain the other.

Example: Analysis of a Square Wave

Let

$$\Psi(x) = +1\,, \textbf{ for } x \in [0, \frac{\ell}{2}] \qquad \Psi(x) = -1\,, \textbf{ for } x \in [\frac{\ell}{2}, \ell]\,.$$

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The Fourier coeffients are easily calculated.

$$A_n = \frac{2}{\ell}\int_0^\ell \Psi\cdot\Psi_n\,dx = \frac{2}{\ell}\left(\int_0^{\ell/2}\sin\frac{n\pi}{\ell}x\,dx - \int_{\ell/2}^{\ell}\sin\frac{n\pi}{\ell}x\,dx\right)\,.$$

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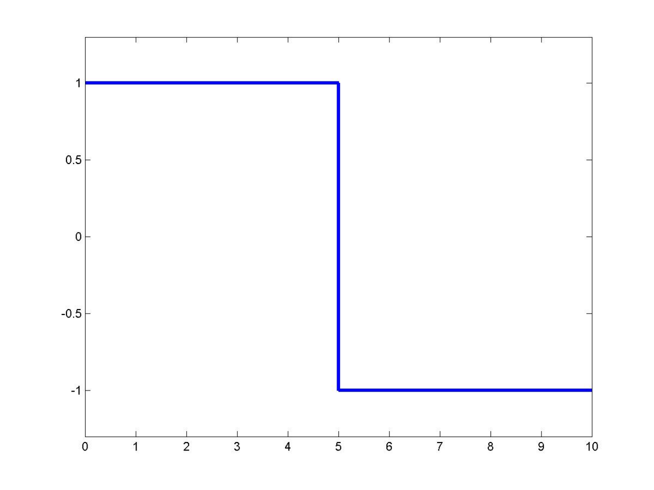
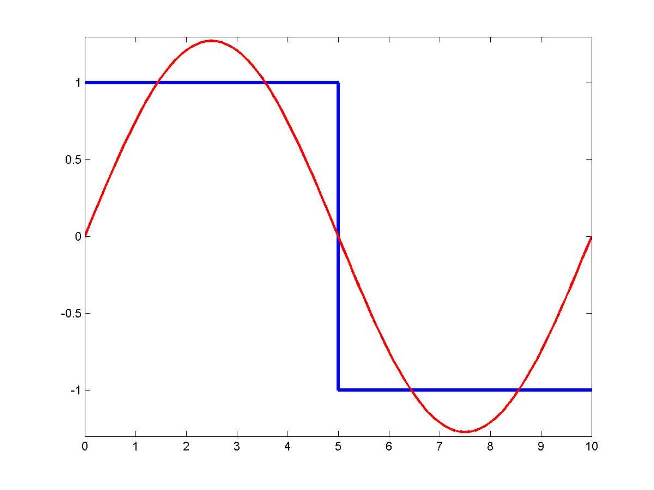
When we work these out, we find that only every fourth coefficient has a nonzero value:

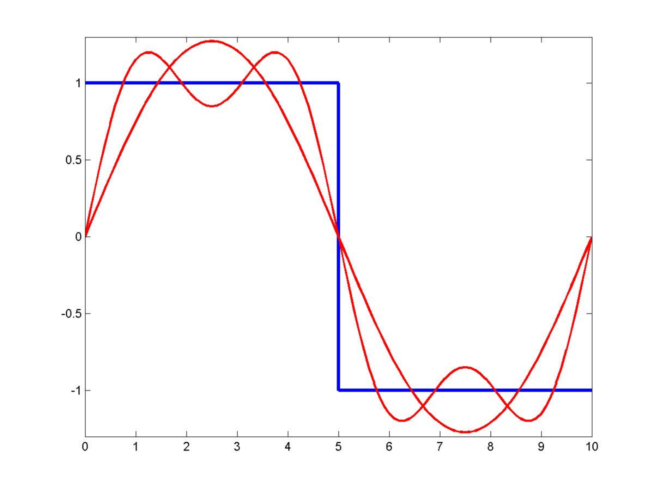
$$A_2 = \frac{1}{2}\left(\frac{8}{\pi}\right)\,, \quad A_6 = \frac{1}{6}\left(\frac{8}{\pi}\right)\,, \quad A_{10} = \frac{1}{10}\left(\frac{8}{\pi}\right)\,, \quad \textbf{etc.}$$



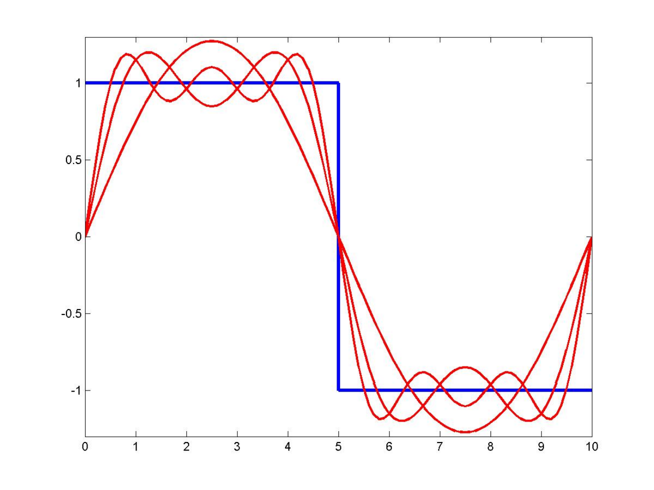
The square wave function ($\ell = 10$).



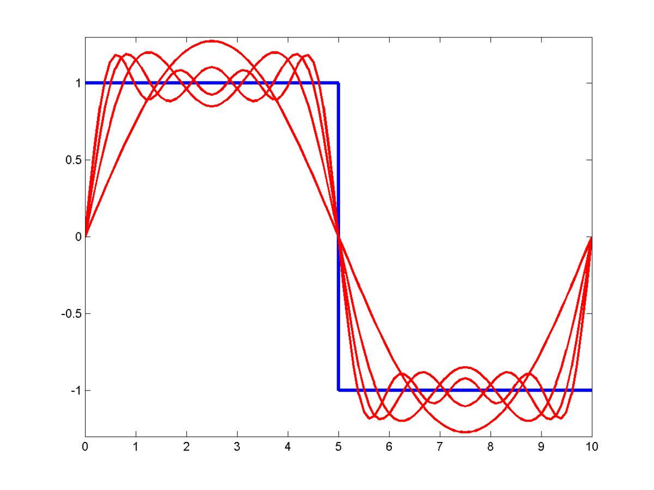
The square wave function. First coefficient.



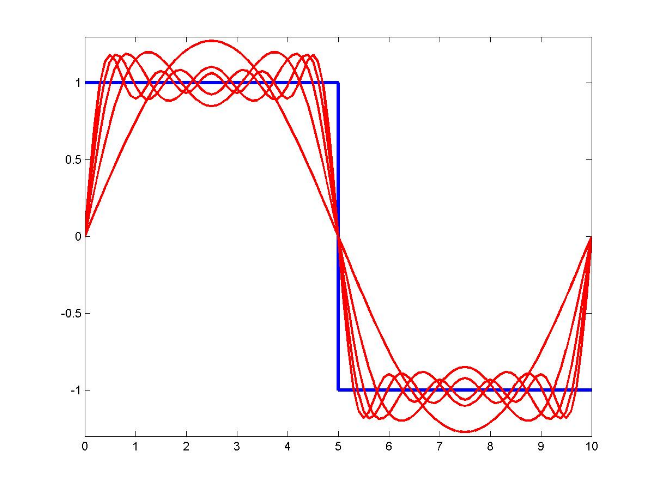
The square wave function. First two coefficients.



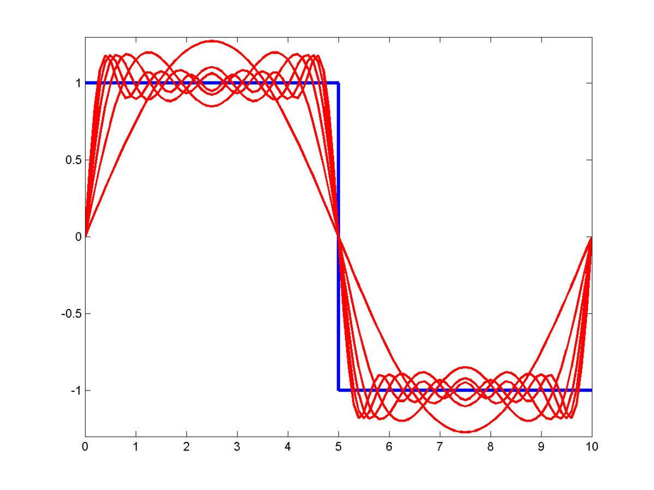
The square wave function. First three coefficients.



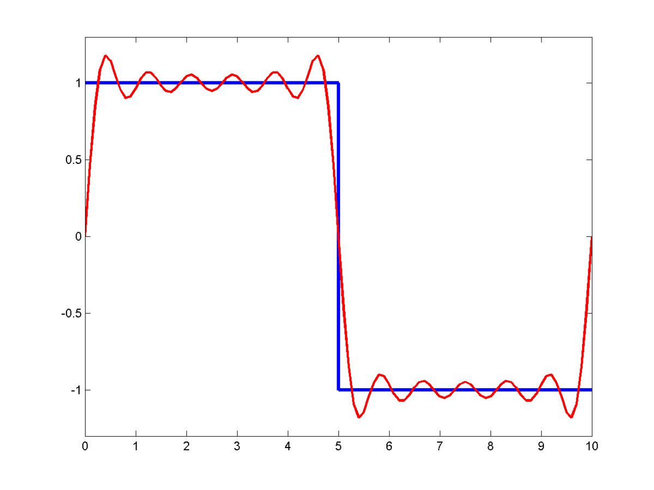
The square wave function. First four coefficients.



The square wave function. First five coefficients.

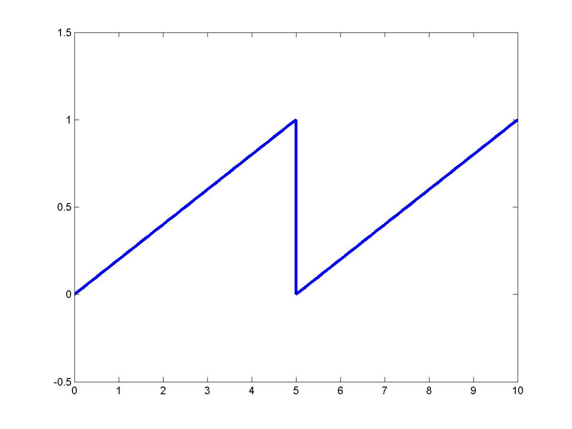


The square wave function. First six coefficients.



First six coefficients. Note Gibbs Phenomenon.

Exercise: Analysis of a Sawtooth Wave



Find the Fourier coefficients of the sawtooth function.

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At the initial time,

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Also, because of the chosen form of solution,

$$\Phi_t(x,0) = 0\,.$$

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This gives us the values of the coefficients:

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$$\Phi(x,t) = \sum_{n=1}^{\infty} A_n \Psi_n(x) \cos \omega_n t \,,$$

with coefficients that are now known.

The eigenfunctions and eigenvalues are defined by

$$\Psi_n(x) \equiv \sin k_n x \,, \qquad k_n = n\pi/\ell \,, \qquad \omega_n = c\, k_n \,.$$

End of Part 1