The Spectral Method (MAPH 40260)

Part 1: Spectral Analysis

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Outline

Introduction

Fourier Analysis

Vibrating String

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Vibrating String

The Gridpoint Method

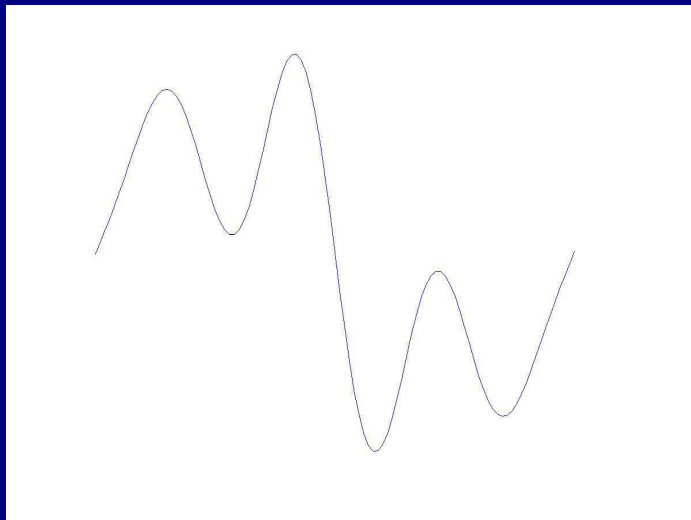
Suppose we have a function of one space coordinate.

For example: the temperature on a line from Galway to Dublin; the pressure around the equator.

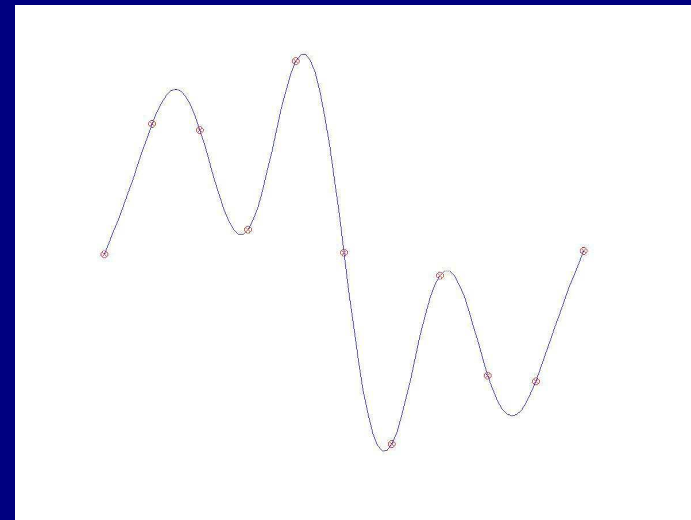
This is an infinite amount of information.

How do we specify the function in a finite way?

There are several answers to this question.



Continuous function of position.



Evalution on a set of grid points.

Grid point values.

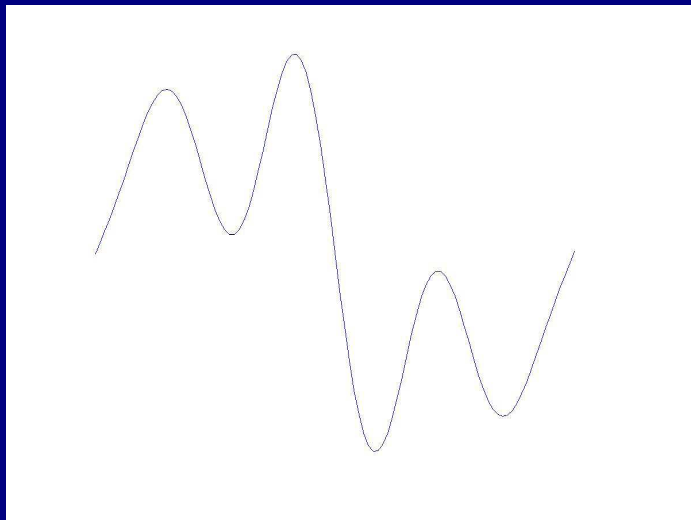
Spectral Analysis

As an alternative to grid point values, we can break the function into different **scales.**

These are called the **spectral components.**

The procedure is called **spectral analysis.**

It is somewhat like splitting sunlight into the various colours of the spectrum.



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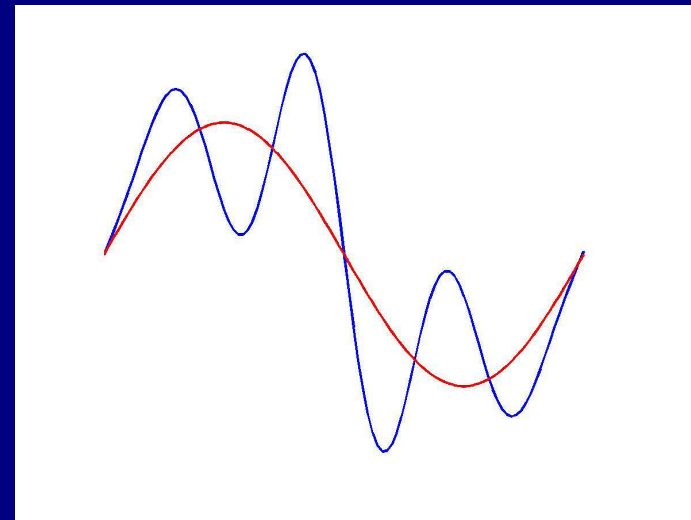
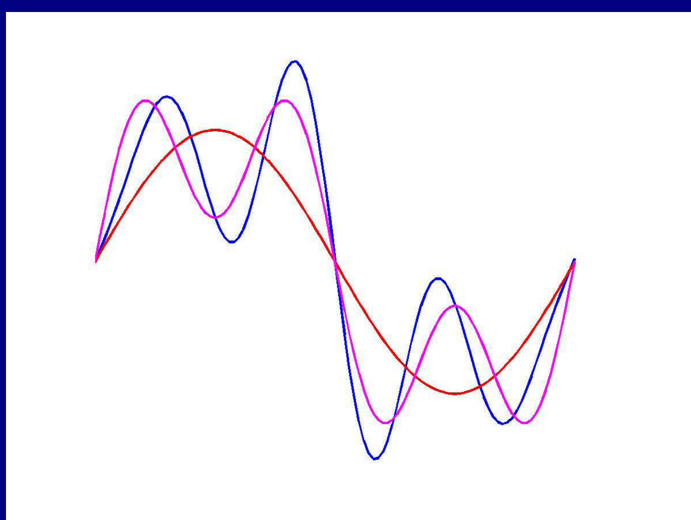
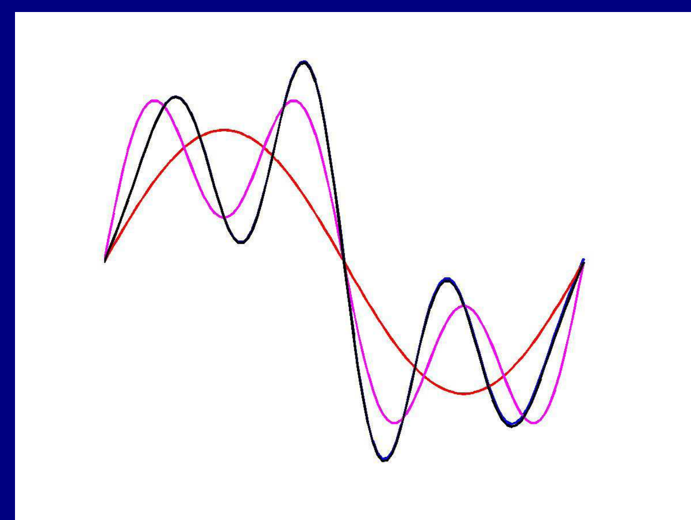


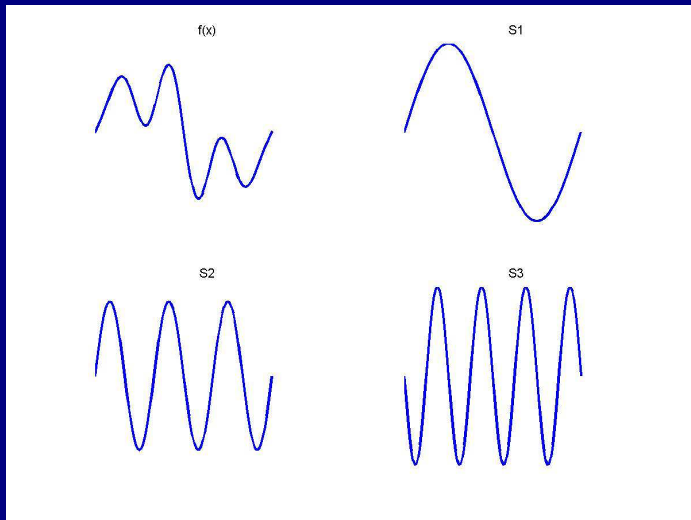
The first spectral component.



The first and second spectral components.



The first, second and third spectral components.



The original function and its three components.

Gridpoint Method

- Discrete representation
- Values at geographical locations
- Easy to understand
- No computation necessary
- Easy to represent graphically.
- Derivatives evaluated by finite differences.

Big drawback: Evaluation of derivatives involves errors.

Grid point values. We have to get derivatives from this set of values.

Spectral Method

- Discrete representation
- Values NOT at geogaphical locations
- Less easy to understand
- Computation of coeficients necessary
- Derivatives evaluated exactly, by analysis.

Big advantage: Evaluation of derivatives is exact.

The most popular spectral components are trigonometric functions.

These functions are easy to evaluate, and to manipulate.

They are also easily differentiated analytically.

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x\,.$$

$$\exp(ix) = \cos x + i\sin x$$

$$\frac{d}{dx}\exp(ix) = i\exp(ix)\,.$$

Outline

Fourier Analysis

We consider a function $f(x)$ on an interval $[0,\ell]$.

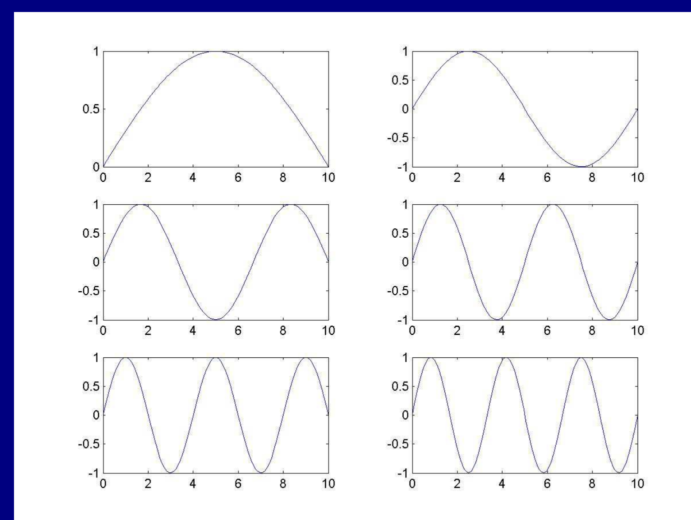
For simplicity, we assume that f vanishes at the ends:

$$f(0) = f(\ell) = 0$$

We note that sinusoidal functions with certain wavelengths also vanish at $x = 0$ and at $x = \ell$:

$$\sin\pi x/\ell\,, \sin 2\pi x/\ell\,, \ldots\,, \sin n\pi x/\ell\,,$$

for all integer values of n.



The first six harmonic components ($\ell = 10$).

Orthogonality

We denote the spectral components by

$$\Psi_n(x) = \sin(n\pi x/\ell)$$

We easily show that

$$\begin{aligned}\int_0^\ell [\Psi_n(x)]^2\,dx &= \int_0^\ell \sin^2\left(\frac{n\pi}{\ell}x\right)\,dx \\ &= \frac{1}{2}\int_0^\ell \left[1-\cos\left(\frac{2n\pi}{\ell}x\right)\right]\,dx \\ &= \left[\frac{x}{2}\right]_0^\ell - \left[\frac{\ell}{4\pi n}\sin\left(\frac{2n\pi}{\ell}x\right)\right]_0^\ell \\ &= \frac{\ell}{2}\,.\end{aligned}$$

Orthogonality

Suppose $m \neq n$.

$$\begin{aligned}&\int_0^\ell \Psi_n(x)\cdot\Psi_m(x)\,dx \\ &= \int_0^\ell \sin\left(\frac{n\pi}{\ell}x\right)\cdot\sin\left(\frac{m\pi}{\ell}x\right)\,dx \\ &= \frac{1}{2}\int_0^\ell \left[\cos\left(\frac{n-m}{\ell}\pi x\right) - \cos\left(\frac{n+m}{\ell}\pi x\right)\right]\,dx \\ &= \frac{1}{2\pi}\left[\frac{\ell}{n-m}\sin\left(\frac{n-m}{\ell}\pi x\right) - \frac{\ell}{n+m}\sin\left(\frac{n+m}{\ell}\pi x\right)\right]_0^\ell \\ &= 0\,.\end{aligned}$$

Orthonormality

We thus have:

$$\int_0^\ell \Psi_n(x)\cdot\Psi_m(x)\,dx = \delta_{mn}\frac{\ell}{2}$$

Now define

$$\tilde\Psi_n(x) = \sqrt{\frac{2}{\ell}}\sin(n\pi x/\ell)$$

We obtain an orthonormal set of functions:

$$\int_0^\ell \tilde\Psi_n(x)\cdot\tilde\Psi_m(x)\,dx = \delta_{mn}\,.$$

Outline

Example: Vibrating String

Imagine a string, stretched between $x = 0$ and $x = \ell$. Let the sideways displacement be $\Phi(x)$.

We suppose the string is fixed at the ends:

$$\Phi(0) = 0 \qquad \Phi(\ell) = 0 .$$

The displacement is governed by the wave equation

$$\frac{\partial^2 \Phi}{\partial t^2} = c^2 \frac{\partial^2 \Phi}{\partial x^2} .$$

We first consider a single component, of frequency ω:

$$\Phi(x, t) = \Psi(x) \exp(i\omega t) .$$

We might also choose $\Phi(x, t) = \Psi(x)\cos(\omega t)$ or $\Phi(x, t) = \Psi(x)\sin(\omega t)$.

Then the wave equation reduces to an o.d.e:

$$\frac{d^2\Psi}{dx^2} + \left(\frac{\omega^2}{c^2}\right)\Psi = 0 .$$

We define the **wavenumber** k as

$$k = \frac{\omega}{c}$$

Then the o.d.e. may be written

$$\frac{d^2\Psi}{dx^2} + k^2\Psi = 0 .$$

Wave speed is wavelength divided by period

$$c = \frac{\lambda}{\tau}$$

Period τ is reciprocal of frequency $\nu = 2\pi\omega$, or

$$\tau = \frac{1}{\nu}, \qquad \text{so that} \qquad c = \lambda\nu .$$

Hence

$$k = \frac{\omega}{c} = \frac{2\pi\nu}{c} = \frac{2\pi}{\lambda} .$$

k is the inverse of the wavelength, with a 2π factor.

The function

$$\Psi(x) = A \sin kx$$

is a solution of the o.d.e., and satisfies the boundary conditions if

$$k\ell = n\pi \qquad \text{or} \qquad k = k_n = n\frac{\pi}{\ell}$$

We thus define the components as

$$\Psi_n(x) = A_n \sin \frac{n\pi}{\ell} x$$

where A_n is the amplitude of the n-th component.

$\Psi_n(x)$ is an **eigenfunction** of the o.d.e. with **eigenvalue** $k_n = n\pi/\ell$.

We now seek a solution expanded in eigenfunctions

$$\Psi = \sum_{n=1}^{\infty} A_n \Psi_n(x) .$$

We can find the coefficients by integration

$$\begin{aligned} \int_0^\ell \Psi_m(x)\Psi(x)\,dx &= \int_0^\ell \Psi_m(x) \left(\sum_{n=1}^{\infty} A_n \Psi_n(x) \right) dx \\ &= \sum_{n=1}^{\infty} A_n \left(\int_0^\ell \Psi_m(x)\Psi_n(x)\,dx \right) \\ &= \sum_{n=1}^{\infty} A_n \frac{\ell}{2} \delta_{mn}\,dx \quad = \quad \frac{\ell}{2} A_m . \end{aligned}$$

Thus,

$$A_m = \frac{2}{\ell} \int_0^\ell \Psi_m(x)\Psi(x)\,dx$$

Duality of the Fourier Transform

The function Ψ can be obtained from the expansion

$$\Psi = \sum_{n=1}^{\infty} A_n \Psi_n(x) .$$

if the coeficients A_n are known.

The coefficients can be found by integration

$$A_m = \frac{2}{\ell} \int_0^\ell \Psi_m(x)\Psi(x)\,dx$$

if the function is known.

There is a duality between $\Psi(x)$, a function in physical space and $\{A_n\}$, the coefficients in wavenumber space. Given either representation, we can obtain the other.

Example: Analysis of a Square Wave

Let

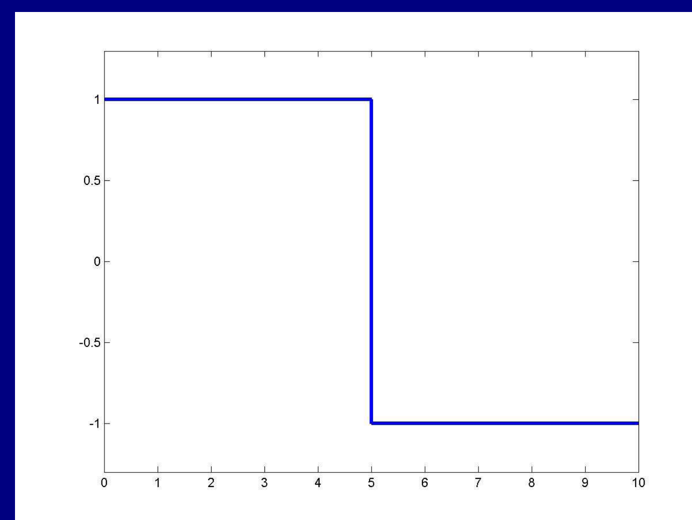
$$\Psi(x) = +1 \,, \text{for } x \in [0, \frac{\ell}{2}] \qquad \Psi(x) = -1 \,, \text{for } x \in [\frac{\ell}{2}, \ell] .$$

The Fourier coeffients are easily calculated.

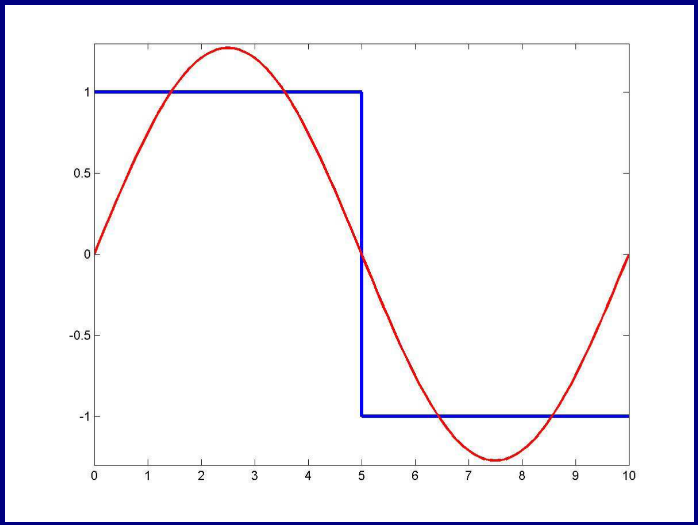
$$A_n = \frac{2}{\ell} \int_0^\ell \Psi \cdot \Psi_n \,dx = \frac{2}{\ell} \left(\int_0^{\ell/2} \sin \frac{n\pi}{\ell} x \,dx - \int_{\ell/2}^{\ell} \sin \frac{n\pi}{\ell} x \,dx \right) .$$

When we work these out, we find that only every fourth coefficient has a nonzero value:

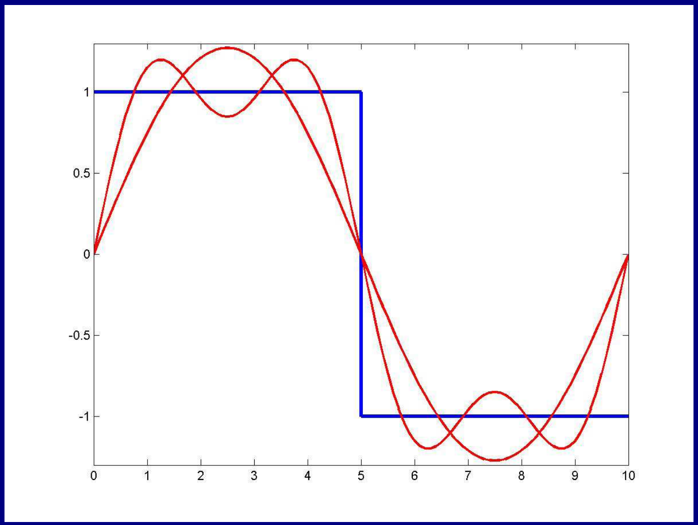
$$A_2 = \frac{1}{2}\left(\frac{8}{\pi}\right) , \quad A_6 = \frac{1}{6}\left(\frac{8}{\pi}\right) , \quad A_{10} = \frac{1}{10}\left(\frac{8}{\pi}\right) , \quad \text{etc.}$$



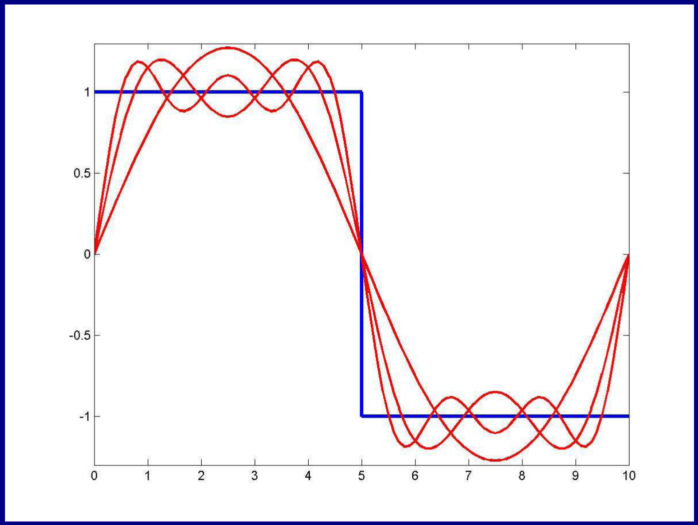
The square wave function ($\ell = 10$).



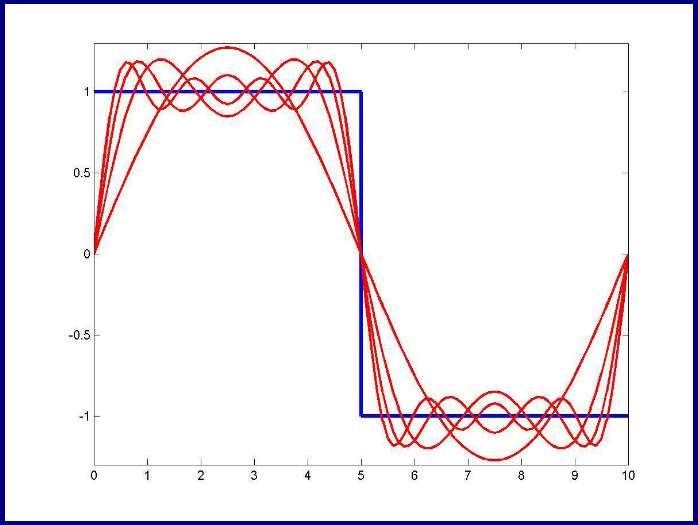
The square wave function. First coefficient.



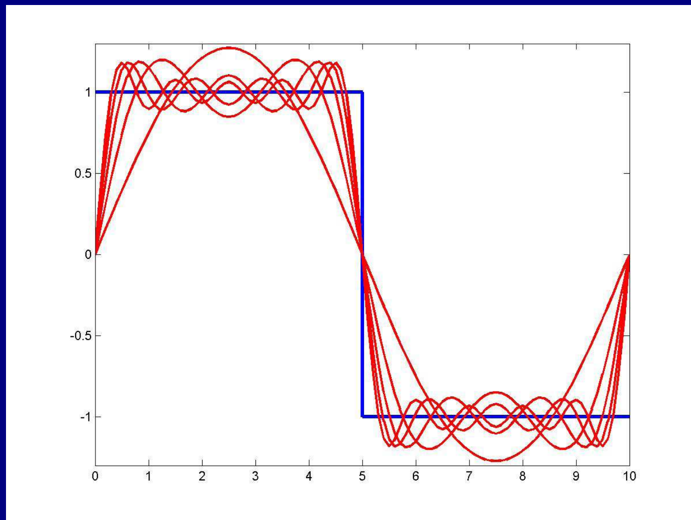
The square wave function. First two coefficients.



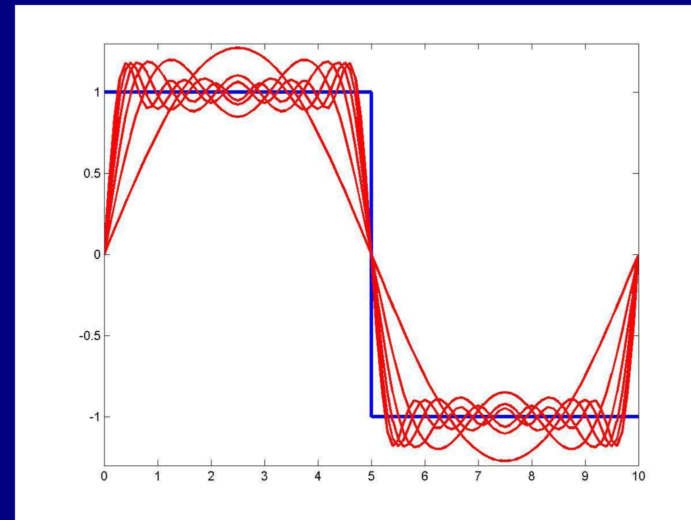
The square wave function. First three coefficients.



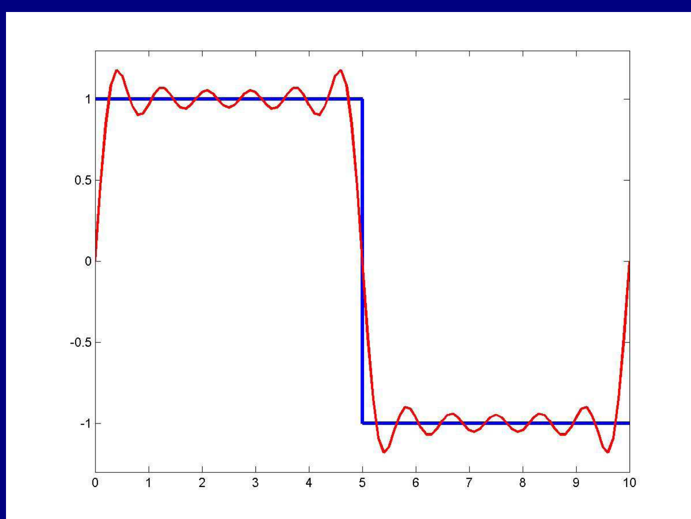
The square wave function. First four coefficients.



The square wave function. First five coefficients.

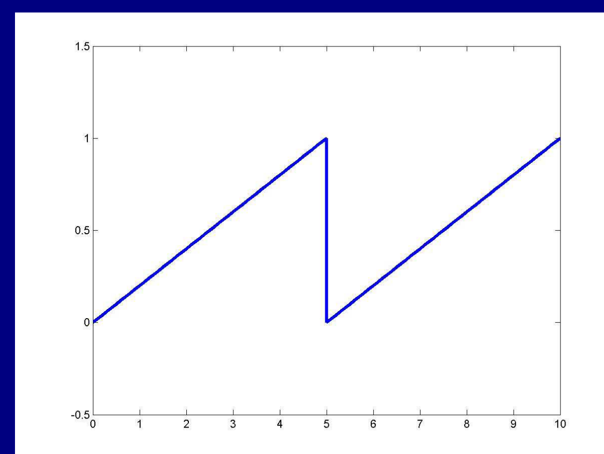


The square wave function. First six coefficients.



First six coefficients. Note Gibbs Phenomenon.

Exercise: Analysis of a Sawtooth Wave



Find the Fourier coefficients of the sawtooth function.

Solution of Wave Equation

We assume simple initial conditions:

$$\Phi(x,0) = \Phi_0(x)\,, \qquad \Phi_t(x,0) = 0\,.$$

We seek a solution of the form

$$\Phi(x,t) = \sum_{n=1}^{\infty} A_n \Psi_n(x) \cos \omega_n t\,.$$

At the initial time,

$$\Phi(x,t) = \Phi_0(x) = \sum_{n=1}^{\infty} A_n \Psi_n(x)\,.$$

Also, because of the chosen form of solution,

$$\Phi_t(x,0) = 0\,.$$

Again,

$$\Phi(x,t) = \Phi_0(x) = \sum_{n=1}^{\infty} A_n \Psi_n(x)\,.$$

This gives us the values of the coefficients:

$$A_n = \frac{2}{\ell} \int_0^{\ell} \Psi_n(x) \Phi_0(x)\, dx$$

The problem is now completely solved:

$$\Phi(x,t) = \sum_{n=1}^{\infty} A_n \Psi_n(x) \cos \omega_n t\,,$$

with coefficients that are now known.

The eigenfunctions and eigenvalues are defined by

$$\Psi_n(x) \equiv \sin k_n x\,, \qquad k_n = n\pi/\ell\,, \qquad \omega_n = c\,k_n\,.$$

End of Part 1