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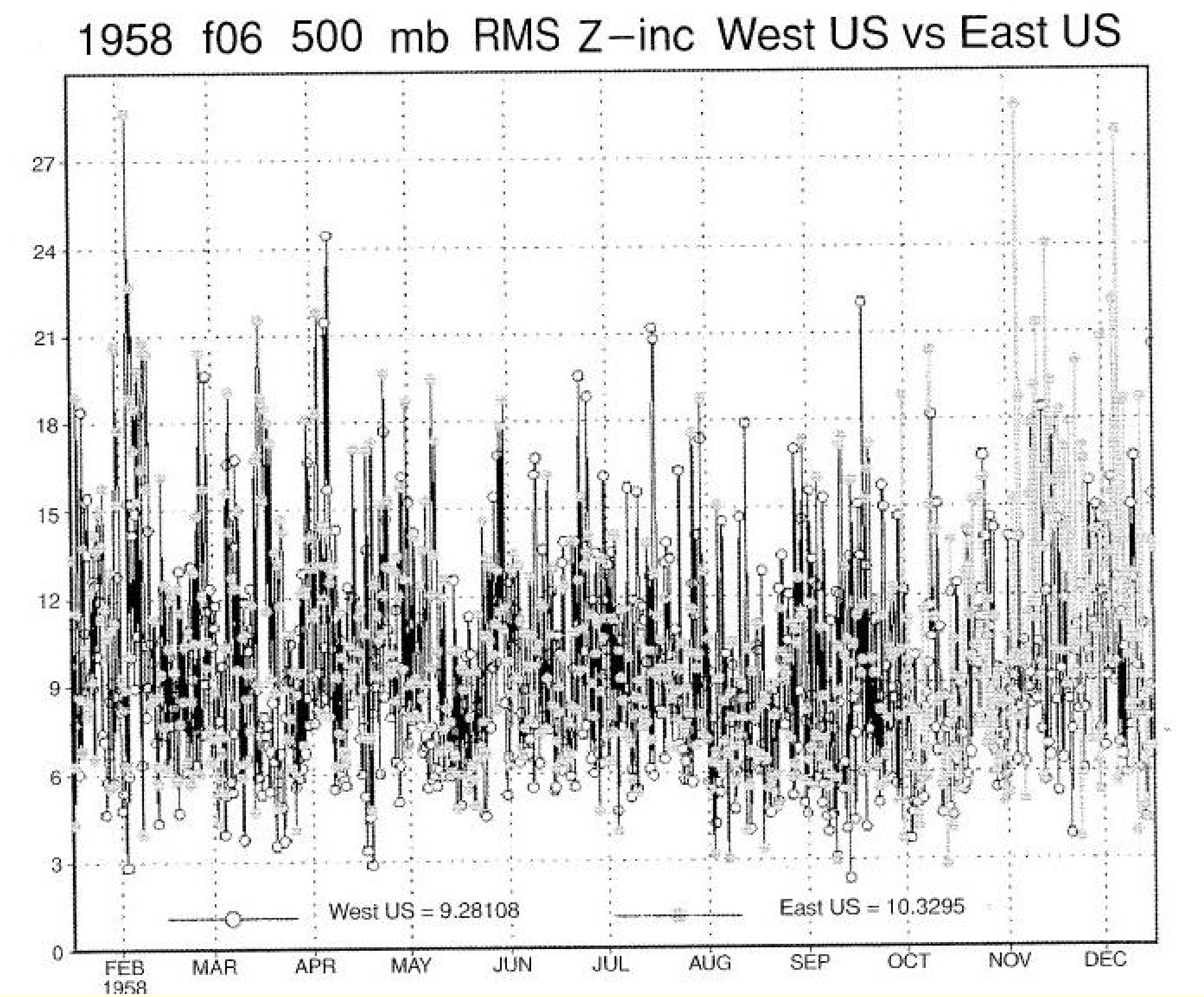
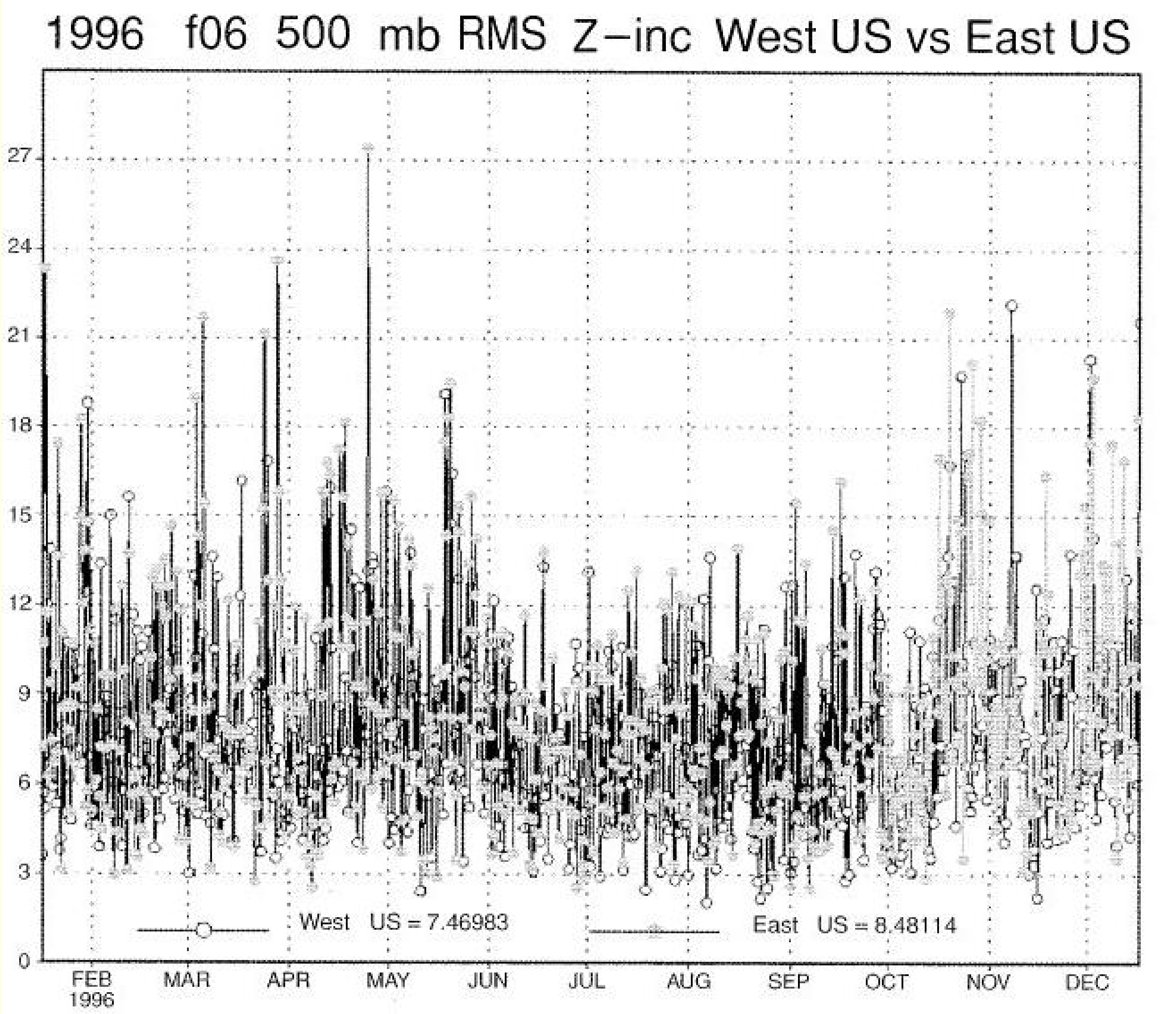
The following figure shows the 6-h forecast errors over the USA from the NCEP/NCAR reanalysis.



Daily variation of the rms increment between the 6-h forecast and the analysis (NCEP-NCAR reanalysis).

1958: $\sigma \approx 10$ m



Daily variation of the rms increment between the 6-h forecast and the analysis (NCEP-NCAR reanalysis).

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Over these four decades the improvements in the observing system in the Northern Hemisphere show a positive impact.

The 6-h forecast errors decrease by about 20%, with the average analysis increment reduced from about 10 m to 8 m.

The most striking result apparent in the error statistics is that the day-to-day variability in the forecast error is about as large as the average error.

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It includes, at least implicitly, the evolution of the forecast error covariance.

Model Error Covariance (Skip)

Let us represent the (nonlinear) model forecast that advances from time t_{i-1} to time t_i by

$$\mathbf{x}^f(t_i) = M_{i-1}\left[\mathbf{x}^a(t_{i-1})\right]$$

Since the model is imperfect, we write

$$\begin{aligned} \mathbf{x}^f(t_i) &= M_{i-1}[\mathbf{x}^t(t_{i-1})] \\ \mathbf{x}^t(t_i) &= M_{i-1}[\mathbf{x}^t(t_{i-1})] - \eta(t_{i-1}) \\ \mathbf{x}^f(t_i) &= \mathbf{x}^t(t_i) + \eta(t_{i-1}) \end{aligned}$$

The model error η is assumed to have zero mean, and covariance matrix $\mathbf{Q}_i = E(\eta_i \eta_i^T)$.

In other words, starting from perfect initial conditions, the forecast error is given by η_i.

(In reality model errors have significant biases, which must be taken into account.)

Note: **I am covering the following material in §5.6 of Eugenia Kalnay's book:**

- **Introductory paragraphs (pp. 175–177)**
- **§5.6.1, to the bottom of page 178**
- **§5.6.3, on 4D-Var**

I am *not* discussing Kalman Filtering in this course.

As this a topic of growing importance, you should read the remaining part of §5.6.1 (pages 179–180) and §5.6.2.

Tangent Linear Model

Consider the solution on the time interval t_i to t_{i+1}.

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The matrix $\mathbf{L}_i$ is the linear tangent model operator

$$[\mathbf{L}_i]_{j,k} = \frac{\partial[M(\mathbf{x}(t_i)]_j}{\partial x_k(t_i)}$$

That is, it is the Jacobian of $M(\mathbf{x})$ with respect to x.

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$$\delta\mathbf{x}(t_{i+1}) = \mathbf{L}_i\delta\mathbf{x}(t_i) + \mathrm{H.O.T.}$$

The Adjoint Model

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For a description of how to develop the computer codes, read Appendix B of Eugenia Kalnay's book.

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The adjoint model advances a perturbation backwards in time, from the final to the initial time.

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The adjoint of $\mathbf{L}_1\mathbf{L}_0$ is $\mathbf{L}_0^T\mathbf{L}_1^T$

The reversal of the order of the terms corresponds to a reversal of time: the operations are preformed backwards.

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& + \frac{1}{2}\sum_{i=0}^{N}\left[H(\mathbf{x}_i) - \mathbf{y}_i^o\right]^T \mathbf{R}_i^{-1}\left[H(\mathbf{x}_i) - \mathbf{y}_i^o\right]
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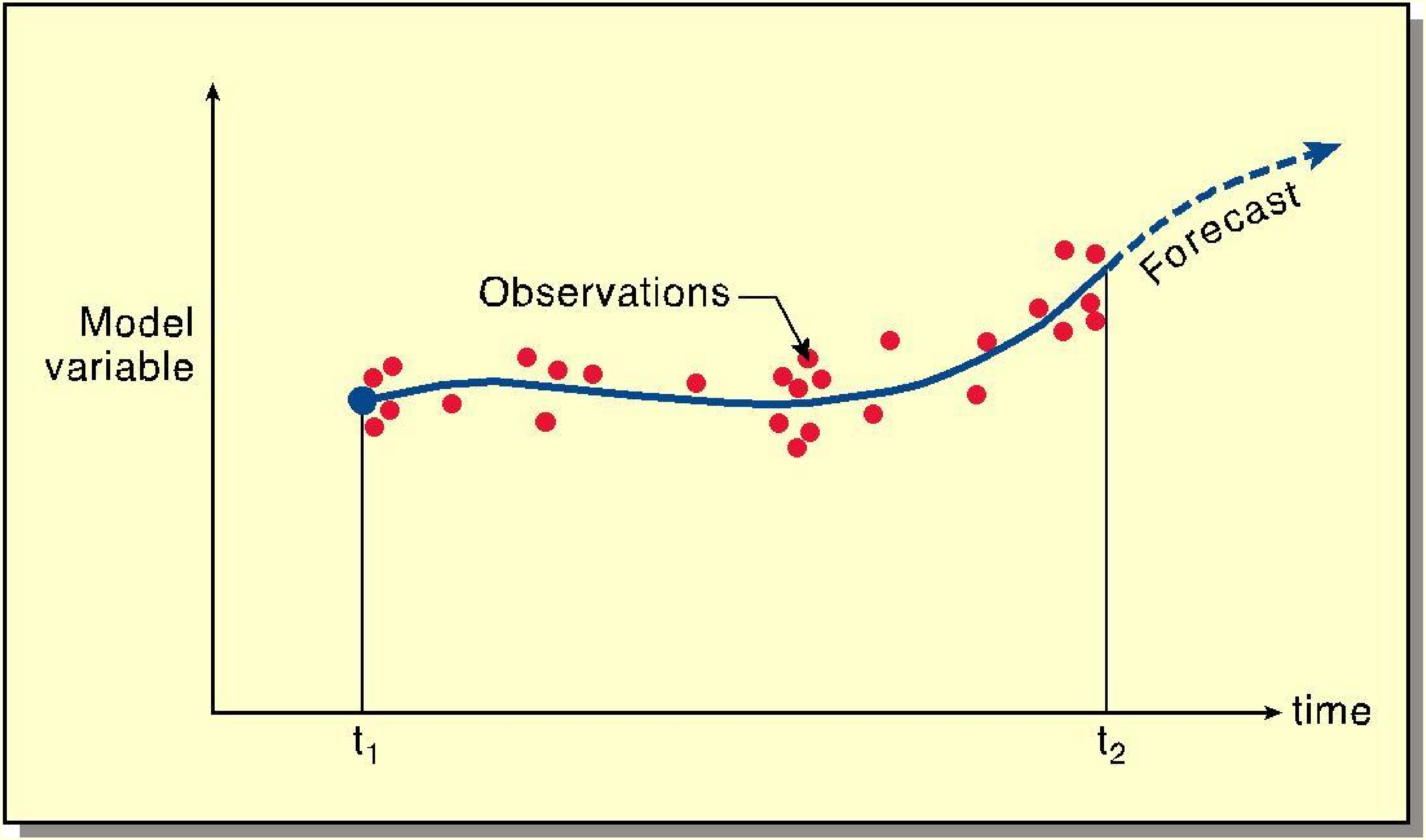
The control variable is the *initial state* of the model $\mathbf{x}\,(t_0)$.



Schematic diagram of four dimensional variational assimilation.

The analysis at the end of the interval is given by the model integration from the solution

$$\mathbf{x}(t_n) = M_{0-n}\left[\mathbf{x}(t_0)\right] = M_{n-1}\left[M_{n-2}\cdots\left[M_1\left[M_0\left[\mathbf{x}(t_0)\right]\right]\cdots\right]\right]$$

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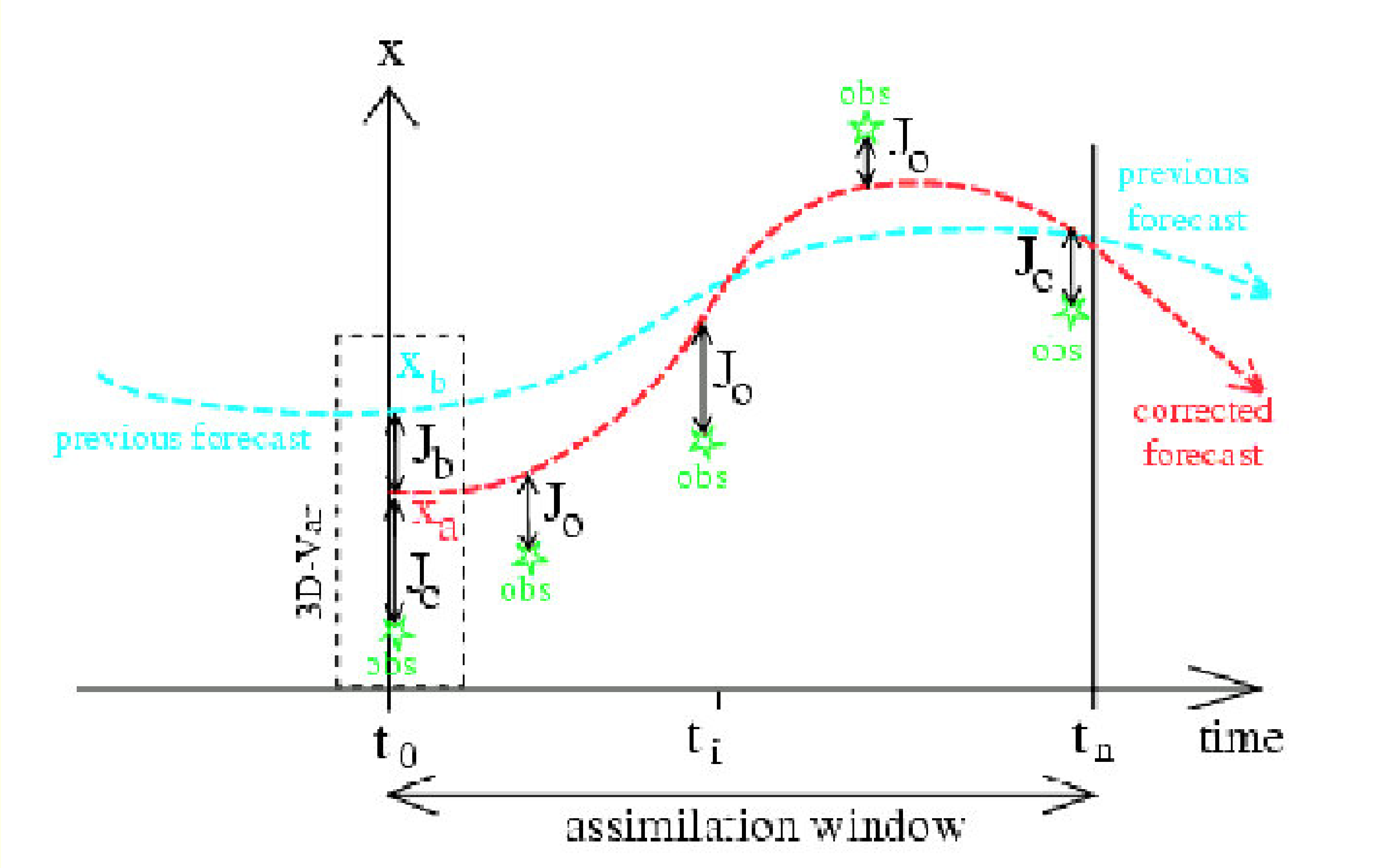
Methods of correcting for a constant model error have been proposed (see references in Kalnay).



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$$\delta J = J[\mathbf{x}(t_0) + \delta\mathbf{x}(t_0)] - J[\mathbf{x}(t_0)] \approx \left[\frac{\partial J}{\partial \mathbf{x}(t_0)}\right]^T \cdot \delta\mathbf{x}(t_0)$$

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We need this because iterative minimization schemes require the estimation of the gradient of the cost function.

In the simplest scheme, the steepest descent method, the change in the control variable after each iteration is chosen to be opposite to the gradient

$$\delta \mathbf{x}(t_0) = -a \nabla_{\mathrm{x}(t_0)} J = -a \, \partial J / \partial \mathbf{x}(t_0) \, .$$

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More efficient methods, such as the conjugate gradient or quasi-Newton method, also require the use of the gradient.

In the simplest scheme, the steepest descent method, the change in the control variable after each iteration is chosen to be opposite to the gradient

$$\delta\mathbf{x}(t_0) = -a\nabla_{\mathrm{x}(t_0)}J = -a\,\partial J/\partial\mathbf{x}(t_0)\,.$$

where a is chosen empirically.

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Thus, in order to solve this minimization problem efficiently, we need to be able to compute the gradient of J with respect to the elements of the control variable.

$\star \quad \star \quad \star$

Lemma I:

Given a symmetric matrix A and a functional $J = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x}$**, the gradient is given by**

$$\frac{\partial J}{\partial \mathbf{x}} = \mathbf{A}\mathbf{x}\,.$$

(we proved this already).

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Lemma II:

If $J = \mathbf{y}^T\mathbf{A}\mathbf{y}$, and $\mathbf{y} = \mathbf{y}(\mathbf{x})$, then

$$\frac{\partial J}{\partial \mathbf{x}} = \left[\frac{\partial \mathbf{y}}{\partial \mathbf{x}}\right]^T \frac{\partial J}{\partial \mathbf{y}} = \left[\frac{\partial \mathbf{y}}{\partial \mathbf{x}}\right]^T \mathbf{A}\mathbf{y}$$

where $[\partial\mathbf{y}/\partial\mathbf{x}]_{k,l} = \partial y_k/\partial x_l$ is a matrix.

Proof of Lemma II:

Consider $J = J(y_1, \ldots, y_n)$ **where** $y_i = y_i(x_1, \ldots, x_n)$**.**

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Q.E.D.

Conclusion of the foregoing

$J = J_b + J_o$

We can write the cost function J as a sum of the background error term and the observation error term

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First, we require the gradient, with respect to $\mathbf{x}(t_0)$, of the background component of the cost function

$$J_b = \frac{1}{2}[\mathbf{x}(t_0) - \mathbf{x}^b(t_0)]^T \mathbf{B}_0^{-1}[\mathbf{x}(t_0) - \mathbf{x}^b(t_0)]$$

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We are half-way there (but it is the easy half).

The gradient of the term J_o is more complicated.

The gradient of the second term,

$$J_o = \frac{1}{2}\sum_{i=0}^{N}[H(\mathbf{x}_i) - \mathbf{y}_i^o]^T \mathbf{R}_i^{-1}[H(\mathbf{x}_i) - \mathbf{y}_i^o]$$

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The matrices $\mathbf{H}_i$ and $\mathbf{L}(t_0, t_i)$ are the linearized Jacobians:

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Expanding the linear tangent model operator step by step,

$$\mathbf{H}_i\mathbf{L}(t_0, t_i) = \mathbf{H}_i\prod_{j=i-1}^{0}\mathbf{L}(t_j, t_{j+1}) = \mathbf{H}_i\left[\mathbf{L}_{i-1}\mathbf{L}_{i-2}\cdots\mathbf{L}_1\mathbf{L}_0\right]\,.$$

Recall that

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Defining the innovation $\mathbf{d}_i = [\mathbf{y}_i^o - H(\mathbf{x}_i)]$, this is

$$\frac{\partial J_o}{\partial \mathbf{x}_0} = -\sum_{i=0}^{N} \left[\mathbf{L}_0^T \mathbf{L}_1^T \cdots \ \mathbf{L}_{i-1}^T\right] \mathbf{H}_i^T \mathbf{R}_i^{-1} \mathbf{d}_i$$

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- **Computing the increments $\mathbf{d}_i = -[H(\mathbf{x}_i) - \mathbf{y}_i^o]$ at the observation times t_i during a forward integration**
- **Multiplying them by $\mathbf{H}_i^T \mathbf{R}_i^{-1}$**
- **Integrating these weighted increments backward to the initial time using the adjoint model.**

Since parts of the backward adjoint integration are common to several time intervals, the summation

$$\sum_{i=0}^{N} \mathbf{L}(t_i, t_0)^T \mathbf{H}_i^T \mathbf{R}_i^{-1} \left[H(\mathbf{x}_i) - \mathbf{y}_i^o \right]$$

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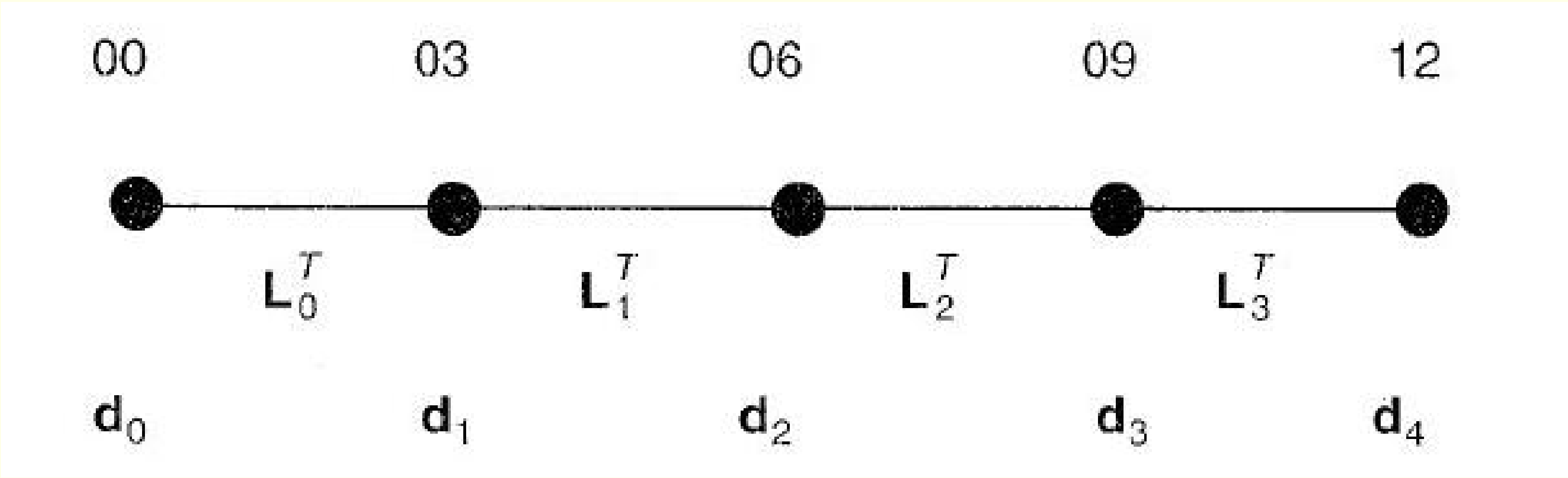

Schematic of the computation of the gradient of the observational cost function for a period of 12 h, with observations every 3 hours.

We compute, during the forward integration, the weighted negative observation increments

$$\overline{\mathbf{d}}_i = \mathbf{H}_i^T \mathbf{R}_i^{-1} [H(\mathbf{x}_i) - \mathbf{y}_i^o] = -\mathbf{H}_i^T \mathbf{R}_i^{-1} \mathbf{d}_i \, .$$

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The adjoint model $\mathbf{L}^T(t_i, t_{i-1}) = \mathbf{L}_{i-1}^T$ applied to a vector $\overline{\mathbf{d}}_\mathbf{i}$ "converts" it from time t_i to time t_{i-1}.

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Recall the equation

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This can be written

$$\frac{\partial J_o}{\partial \mathbf{x}_0} = \Big[\overline{\mathbf{d}}_0 + \mathbf{L}_0^T \overline{\mathbf{d}}_1 + \mathbf{L}_0^T \mathbf{L}_1^T \overline{\mathbf{d}}_2 + \\ \mathbf{L}_0 \mathbf{L}_1^T \mathbf{L}_2^T \overline{\mathbf{d}}_3 + \mathbf{L}_0 \mathbf{L}_1 \mathbf{L}_2^T \mathbf{L}_3^T \overline{\mathbf{d}}_4 \Big]$$

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Thus, we can write the gradient of J as

$$\frac{\partial J_o}{\partial \mathbf{x}_o} = \overline{\mathbf{d}}_o + \mathbf{L}_0^T \left\{\overline{\mathbf{d}}_1 + \mathbf{L}_1^T \left[\overline{\mathbf{d}}_2 + \mathbf{L}_2^T \left(\overline{\mathbf{d}}_3 + \mathbf{L}_3^T \overline{\mathbf{d}}_4\right)\right]\right\}$$

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- Integrate the full model forward, computing and storing the increments $\mathbf{d}_i$ at the observation times t_i.
- Integrate the adjoint model backwards, accumulating the terms $\overline{\mathbf{d}}_i = -\mathbf{H}_i^T\mathbf{R}_i^{-1}\mathbf{d}_i$, using the adjoint model.

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- **Iterate these forward-backward cycles until convergence.**

Reduced Inner Loops

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\begin{aligned}
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&+ \frac{1}{2}\sum_{i=0}^{N}\left[\mathbf{H}_i\mathbf{L}(t_0,t_i)\delta\mathbf{x}_0 - \mathbf{d}_i^o\right]^T\mathbf{R}_i^{-1}\left[\mathbf{H}_i\mathbf{L}(t_0,t_i)\delta\mathbf{x}_0 - \mathbf{d}_i^o\right].
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$$

With the incremental formulation, we introduce a *"simplification operator"* S.

This converts the variables to a lower dimensional space than that of the original model variables x:

$$\delta\mathbf{w} = \mathbf{S}\delta\mathbf{x}$$

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$$
\begin{aligned}
J(\delta\mathbf{x}_0) = {} & \frac{1}{2}(\delta\mathbf{x}_0)^T\mathbf{B}_0^{-1}(\delta\mathbf{x}_0) \\
& + \frac{1}{2}\sum_{i=0}^{N}\left[\mathbf{H}_i\mathbf{L}(t_0,t_i)\delta\mathbf{x}_0 - \mathbf{d}_i^o\right]^T\mathbf{R}_i^{-1}\left[\mathbf{H}_i\mathbf{L}(t_0,t_i)\delta\mathbf{x}_0 - \mathbf{d}_i^o\right].
\end{aligned}
$$

With the incremental formulation, we introduce a *"simplification operator"* S.

This converts the variables to a lower dimensional space than that of the original model variables x:

$$\delta\mathbf{w} = \mathbf{S}\delta\mathbf{x}$$

Typically, S is a projection to a lower dimensional subspace of the total model space.

A number of iterations are now executed in the reduced space. These are called the *"inner loops"*.

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Note that the complete documentation of the ECMWF variational assimilation system is available at:

`http://www.ecmwf.int`

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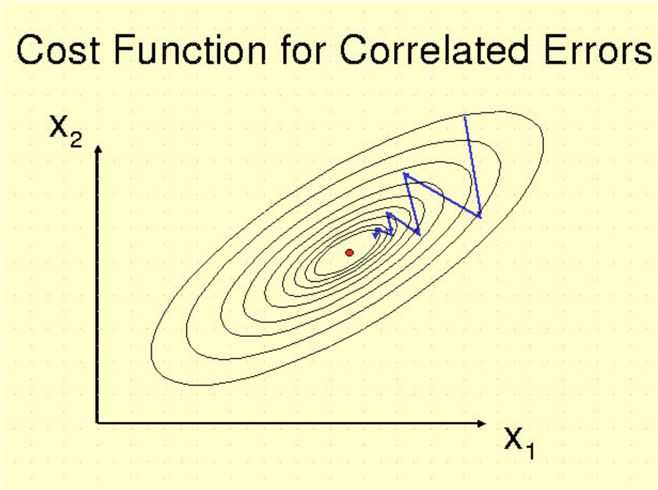

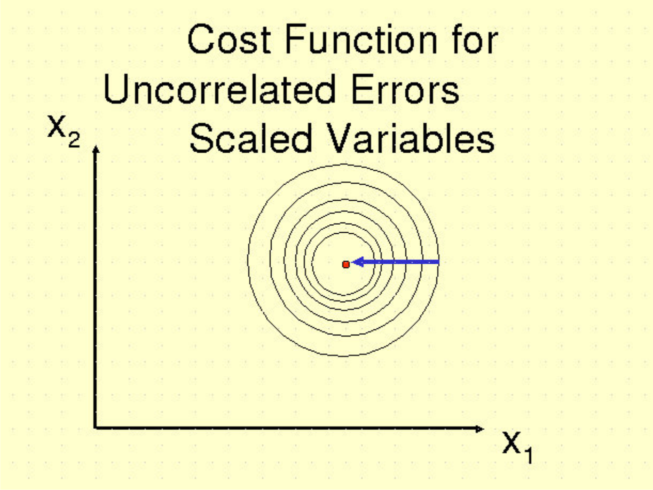

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This means that *implicitly*, 4D-Var is able to evolve the forecast error covariance from B_0 *to the final time*.

Unfortunately, this implicit covariance is not available at the end of the cycle, and neither is the new analysis error covariance.

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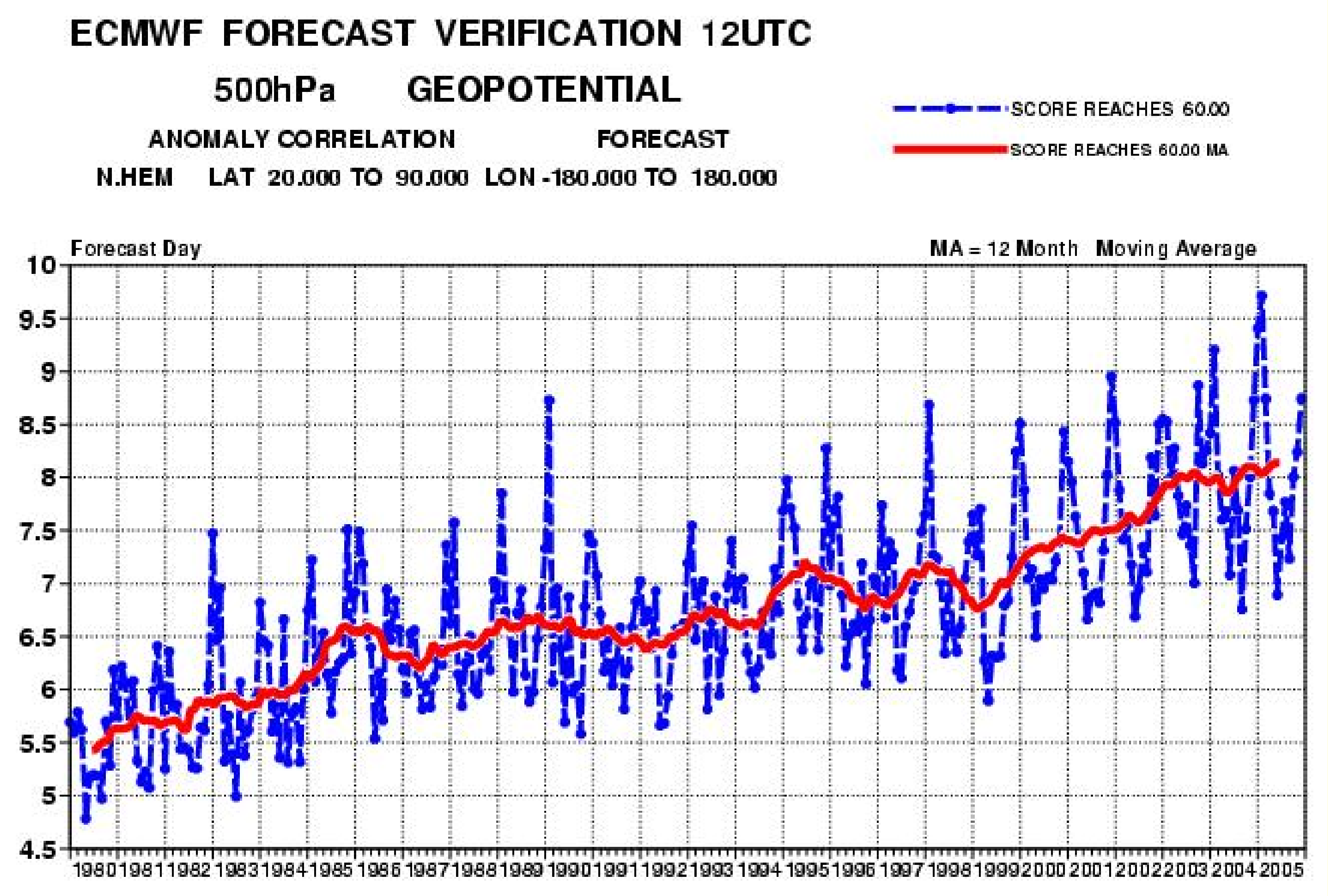
The following figure shows that implementation of 4D-Var has resulted in improved forecast scores.



ECMWF Forecast Verification

WGNE List of Operational Global Numerical Weather Prediction Systems (as of January 2006)

Forecast Centre (Country)	Computer (Peak in TFlop/s)	High resolution Model (FC Range in days)	Ensemble Model (FC Range in days)	Type of Data Assimilation
ECMWF (Europe)	IBM p690, 2x68 nodes (20)	T_L511 L60 (10)	T_L255 L40; M51 (10)	4D-VAR (T_L159)
Met Office (UK)	NEC SX6, 34 nodes NEC SX8 16 nodes (4)	~40km L50 (6)	~90km L38; M24 (3)	4D-Var (~120km)
Météo France (France)	Fujitsu VPP5000 (1.2)	T_L358 (C2.4) L41 (3)	T_L358(C2.4) L41; M11 (2.5)	4D-Var (T_L149)
DWD (Germany)	IBM p575; 2x52 nodes (2x3.1)	40 km L40 (7)	No EPS	3D-OI
HMC (Russia)	Itanium 4x4; Xeon 2x4 (0.10; 0.028)	T85 L31 (10); 0.72°x0.9° L28 (10)	No EPS	3D-OI
NCEP (USA)	IBM p655 (Cluster 1600) (7.8)	T382 L64 (7.5) T190 L64 (16)	T126 L28; M45 (16)	3D-Var (T382)
Navy/NRL (USA)	SGI O3000 (1024 proc) (1.125)	T239 L30 (6)	T119 L30; M10 (10)	3D-Var
CMC (Canada)	IBM p690, 108 nodes (4.3)	0.9°x0.9° L28 (10)	SEF (T_L149); GEM (1.2°); M16 (16)	Det: 4D-Var (1.5°, 0.9°) EPS: EnKF M96 (1.2°)
CPTEC/INPE (Brazil)	NEC SX6, 12 nodes (0.768)	T126L28, T213 L42 (15, 7)	T126 L28; M15 (15)	3D-Var
JMA (Japan)	Hitachi SR8000-E1, 80 nodes (0.768)	T_L319 L40 (9)	T106 L40; M25 (9)	4D-Var (T63)
CMA (China)	SW1; IBM P655/P690 (0.384; 7)	T213 L31 (10)	T106 L19; M33 (10)	3D-OI
KMA (Korea)	Cray X1E-8/1024-L (18.4)	T426 L40 (10)	T106 L30; M17 (8)	3D-Var
NCMRWF (India)	Cray SV1 24 processor (0.028)	T170 L28 (5)	No EPS	3D-VAR
BMRC (Australia)	NEC SX6, 28 nodes (1.792)	T_L239 L29 (10)	T_L119 L19; M33 (10)	3D-OI

Operational global NWP systems (January, 2006)

WGNE Overview of Plans at NWP Centres with Global Forecasting Systems

Part II: Global Modelling

c) Global Data Assimilation Scheme (Type, resolution, number of layers)

Forecast Centre (Country)	2006	2007	2008	2009	2010	2011
ECMWF (Europe)	4D-Var; T_L799 with T255 final inner loop; L91	4D-Var; T_L799 with T255 final inner loop; L91	4D-Var; T_L799 with T255 final inner loop; L91	4D-Var; T_L799 with T255 final inner loop; L91	?	?
Met Office (UK)	4D-Var; 120 km; L50	4D-Var; 120 km; L70	4D-Var; 120 km; L70	4D-Var; 75 km; L90	4D-Var; 75 km; L90	4D-Var; 75 km; L90
Météo France (France)	4D-Var; T159	4D-Var; T250	4D-Var; T250	4D-Var; T250	4D-Var; T350	4D-Var; T350
DWD (Germany)	OI; 40 km; L40	3D-Var; 40 km; L40	3D-Var; 40 km; L40	ETKF?	ETKF?	ETKF?
HMC (Russia)	OI; 0.9x0.72; L28	OI; 0.9x0.72; L28	3D-Var; 0.9x0.72; L28	?	?	?
NCEP (USA)	3D-Var; T382	Advanced-Var; T511	Advanced-Var; T511	Adv or 4D-Var; 20 km	Adv or 4D-Var; 20 km	4D-Var; 20 km
Navy/NRL (USA)	3D-Var; T239; L30	3D-Var; T239; L30	3D-Var; T319; L36	4D-Var	4D-Var	4D-Var
CMC (Canada)	4D-Var; 1.5°, 35 km; L58	4D-Var; 1.5°, 35 km; L58	4D-Var; 0.9°, 35 km; L80	4D-Var/EnKF?	4D-Var/EnKF?	4D-Var/EnKF?
CPTEC/INPE (Brazil)	3D-Var; 100 km	3D-Var; 60 km	LENKF; 40 km	LENKF; 40 km	LENKF; 40 km	LENKF; 20 km
JMA (Japan)	4D-Var; 120 km; L40	4D-Var; 80 km; L60	4D-Var; 60 km; L60	4D-Var; 60 km; L60	ETKF	ETKF
CMA (China)	NO RESPONSE					
KMA (Korea)	3D-Var; T426; L40	3D-Var; T426; L40	3D-Var; T426; L70	4D-Var? EnKF?	4D-Var? EnKF?	4D-Var? EnKF?
NCMRWF (India)						
BMRC (Australia)	3D-OI	Met Office 4D-VAR under ACCESS (?)	?	?	?	?

Planned future global data assimilation systems.

End of §5.6