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This method is of growing popularity and is now in use in several major NWP centres.

Variational assimilation has been shown to yield significant improvements in the quality of numerical forecasts.

It has also been invaluable for re-analysis:
The ERA-40 Project at ECMWF was carried out using the 3D-Var system.

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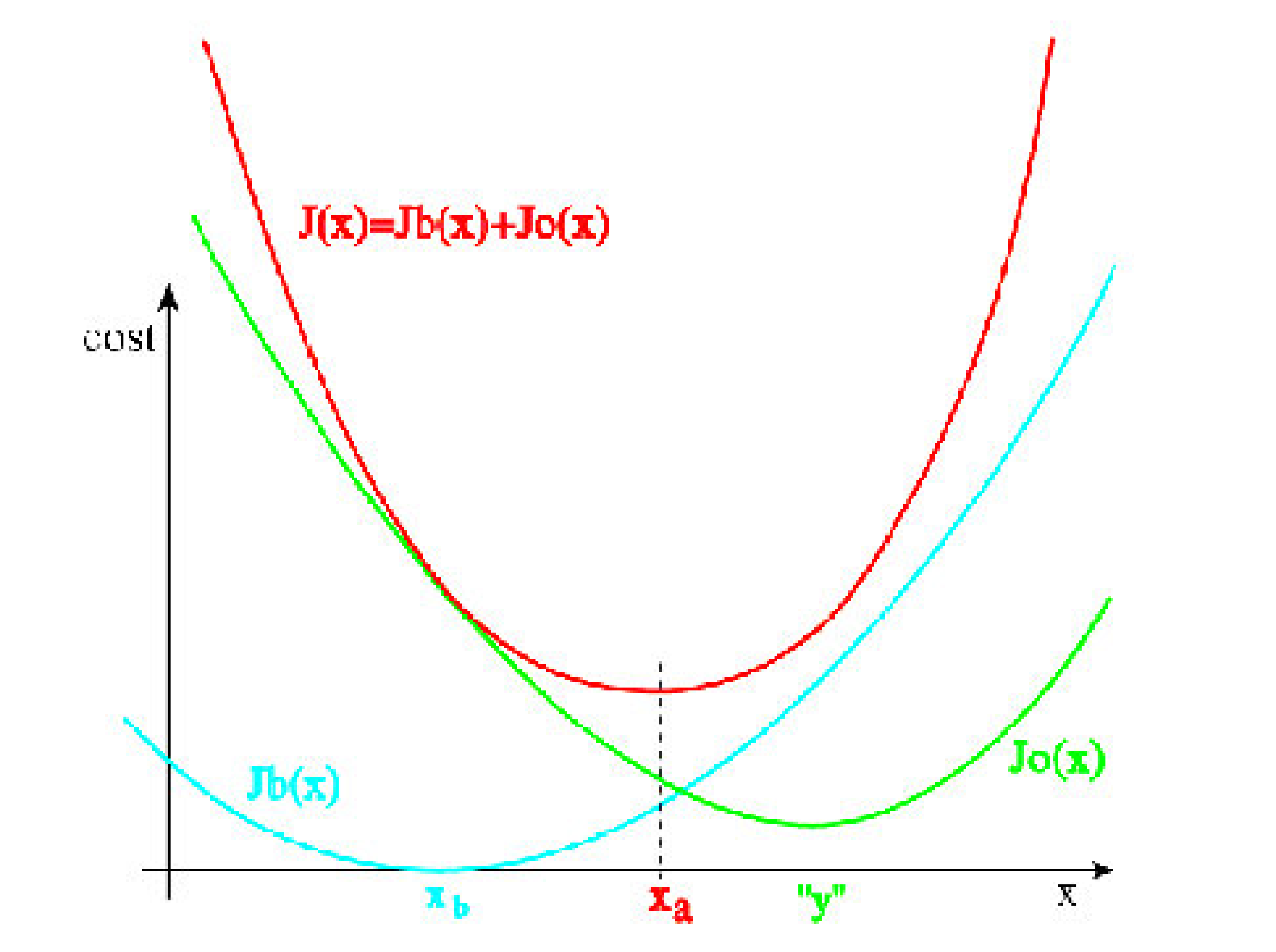
Lorenc (1986) showed that the OI solution is equivalent to a specific variational assimilation problem: Find the optimal analysis $\mathbf{x}_a$ field that minimizes a (scalar) cost function.

The cost function is defined as the (weighted) distance between $\mathbf{x}$ and the background $\mathbf{x}_b$, *plus the (weighted) distance to the observations* $\mathbf{y}_{o,}$:

$$J(\mathbf{x}) = \frac{1}{2}\left\{(\mathbf{x}-\mathbf{x}_b)^T\mathbf{B}^{-1}(\mathbf{x}-\mathbf{x}_b) + [\mathbf{y}_o - H(\mathbf{x})]^T\mathbf{R}^{-1}[\mathbf{y}_o - H(\mathbf{x})]\right\}$$



Schematic representation of the cost function in a simple one-dimensional case. J_b and J_o respectively tend to pull the analysis towards the background $\mathbf{x}_b$ and the observation $\mathbf{y}$.

Again, the cost function is

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The minimum of $J(\mathbf{x})$ is attained for $\mathbf{x} = \mathbf{x}_a$ such that

$$\frac{\partial J}{\partial \mathbf{x}} = \nabla_{\mathbf{x}} J(\mathbf{x}_a) = 0 \qquad\qquad (n \times 1)$$

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Assuming the analysis is close to the truth, we write

$$\mathbf{x} = [\mathbf{x}_b + (\mathbf{x} - \mathbf{x}_b)]$$

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Then we can linearize the observation operator:

$$[\mathbf{y}_o - H(\mathbf{x})] = \mathbf{y}_o - H[\mathbf{x}_b + (\mathbf{x}-\mathbf{x}_b)] = \{\mathbf{y}_o - H(\mathbf{x}_b)\} - \mathbf{H}\cdot(\mathbf{x}-\mathbf{x}_b)$$

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We now substitute this into the cost function:

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The result is:

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2J(\mathbf{x}) = {} & (\mathbf{x}-\mathbf{x}_b)^T\mathbf{B}^{-1}(\mathbf{x}-\mathbf{x}_b) \\
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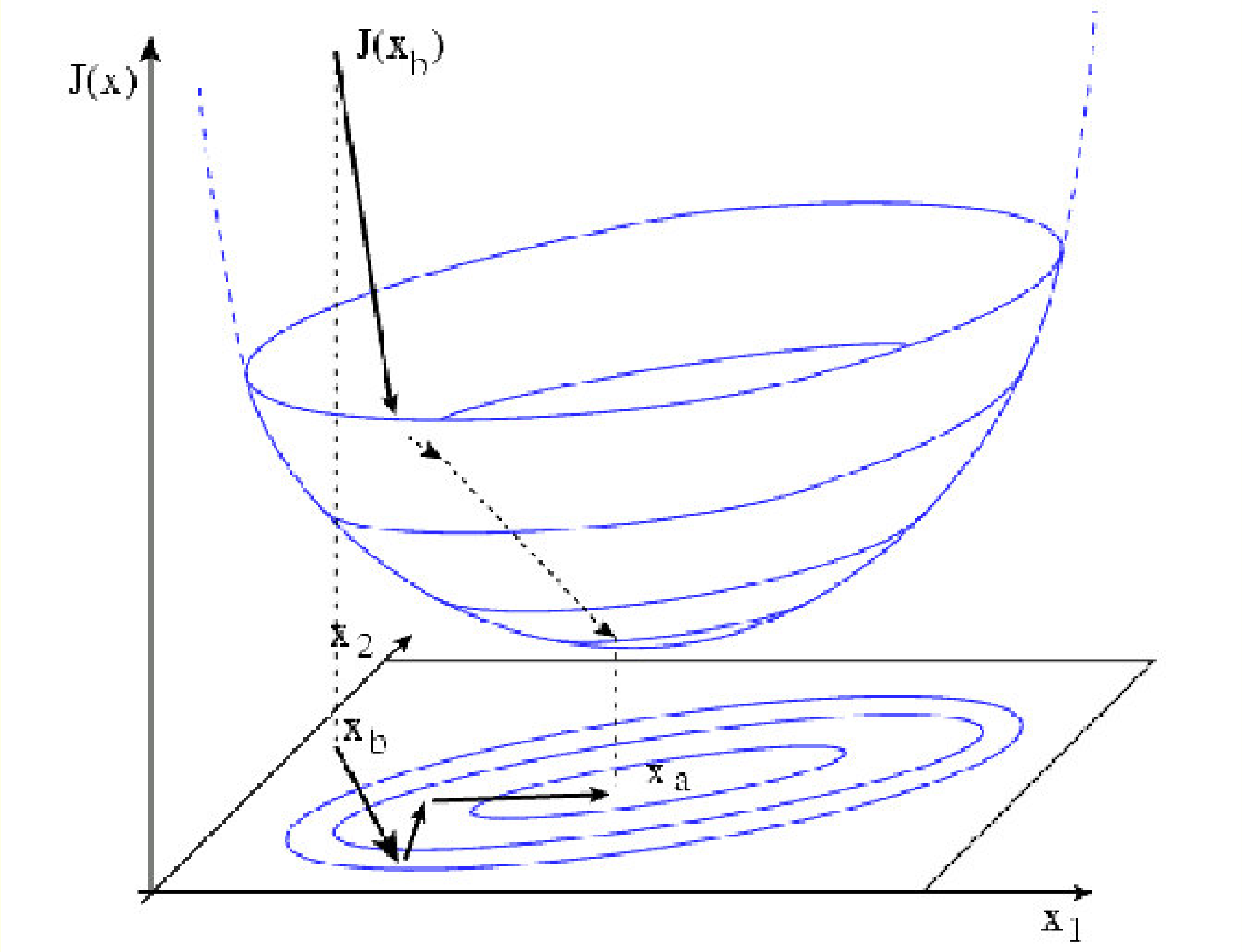
The cost function is a quadratic function of the analysis increments $(\mathbf{x}-\mathbf{x}_b)$.



Schematic of the cost function in two dimensions. The minimum is found by moving down-gradient in discrete steps.

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We use the following Lemma:

Given a quadratic function $F(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{d}^T\mathbf{x} + c$, ***where*** $\mathbf{A}$ ***is a symmetric matrix,*** $\mathbf{d}$ ***is a vector and*** c ***a scalar, the gradient is given by*** $\nabla F(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{d}$.

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So the derivative w.r.t. x_k **is**

$$\frac{\partial F}{\partial x_k} = \frac{1}{2}\sum_j A_{kj}x_j + \frac{1}{2}\sum_j A_{ik}x_i + d_k = \sum_j A_{kj}x_j + d_k$$

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Q.E.D.

Recall the cost function was

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Combining the first two terms, we get

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The gradient of the cost function J with respect to $\mathbf{x}$ is

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We now set $\nabla J(\mathbf{x}_a) = 0$ to ensure that J is a minimum, and obtain an equation for $(\mathbf{x}_a - \mathbf{x}_b)$

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This is the solution of the 3-dimensional variational (3D-Var) analysis problem.

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The equivalence is *not obvious*.

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The equivalence between the minimization of the analysis error variance and the three-dimensional variational cost function approach is an important property.

Conclusion of the foregoing

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For a selection of techniques, see Numerical Recipes, which may be inspected online before purchase.

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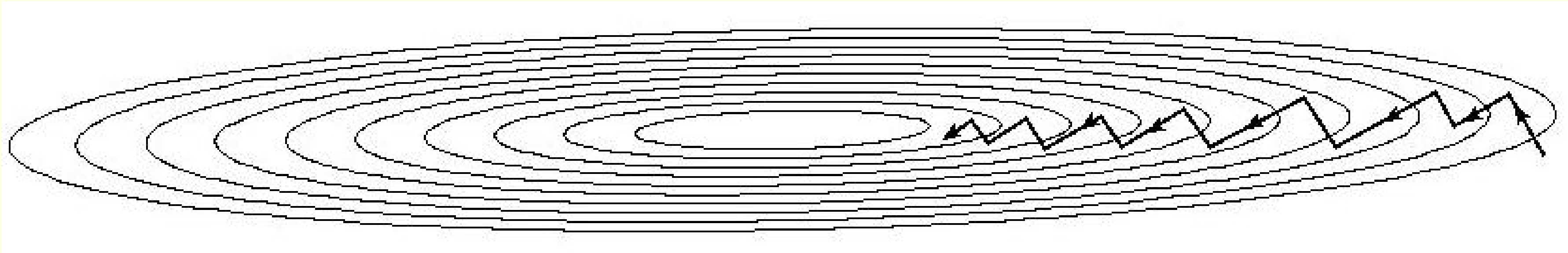

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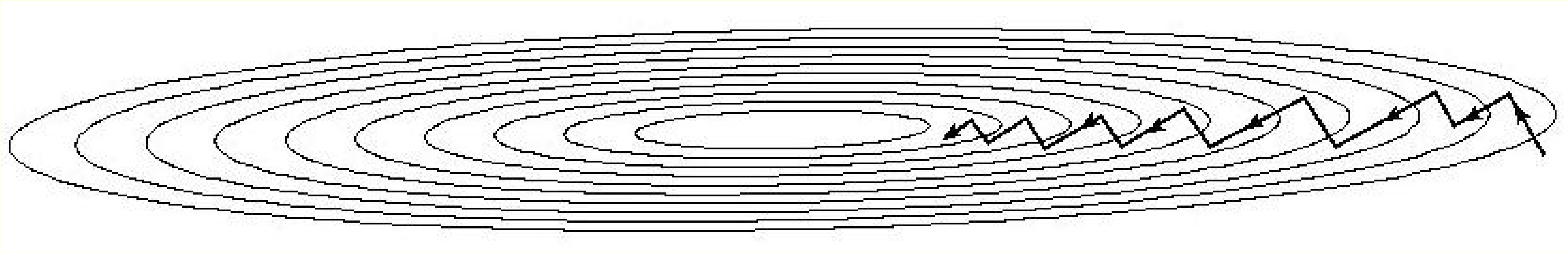

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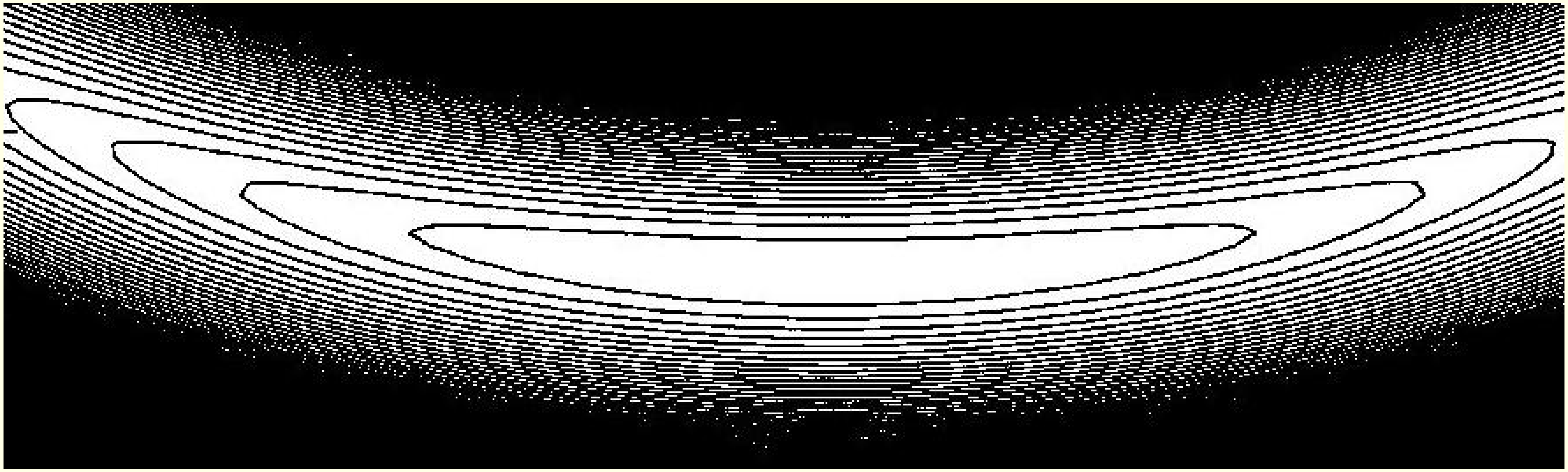
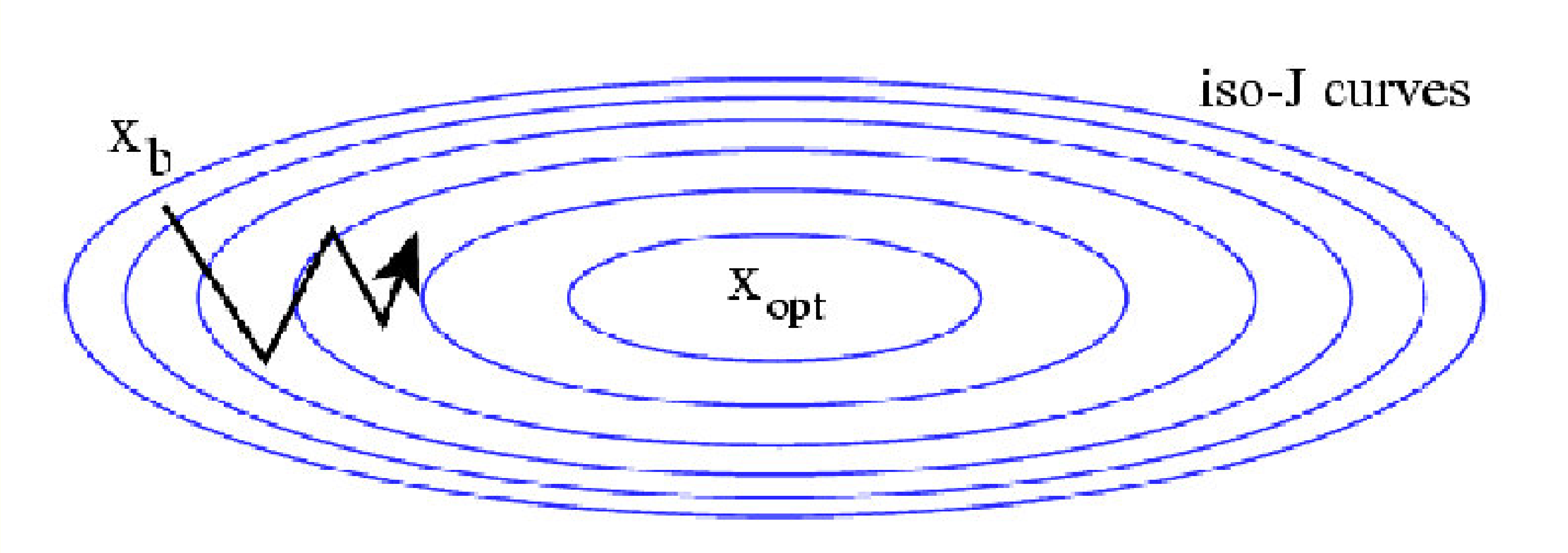
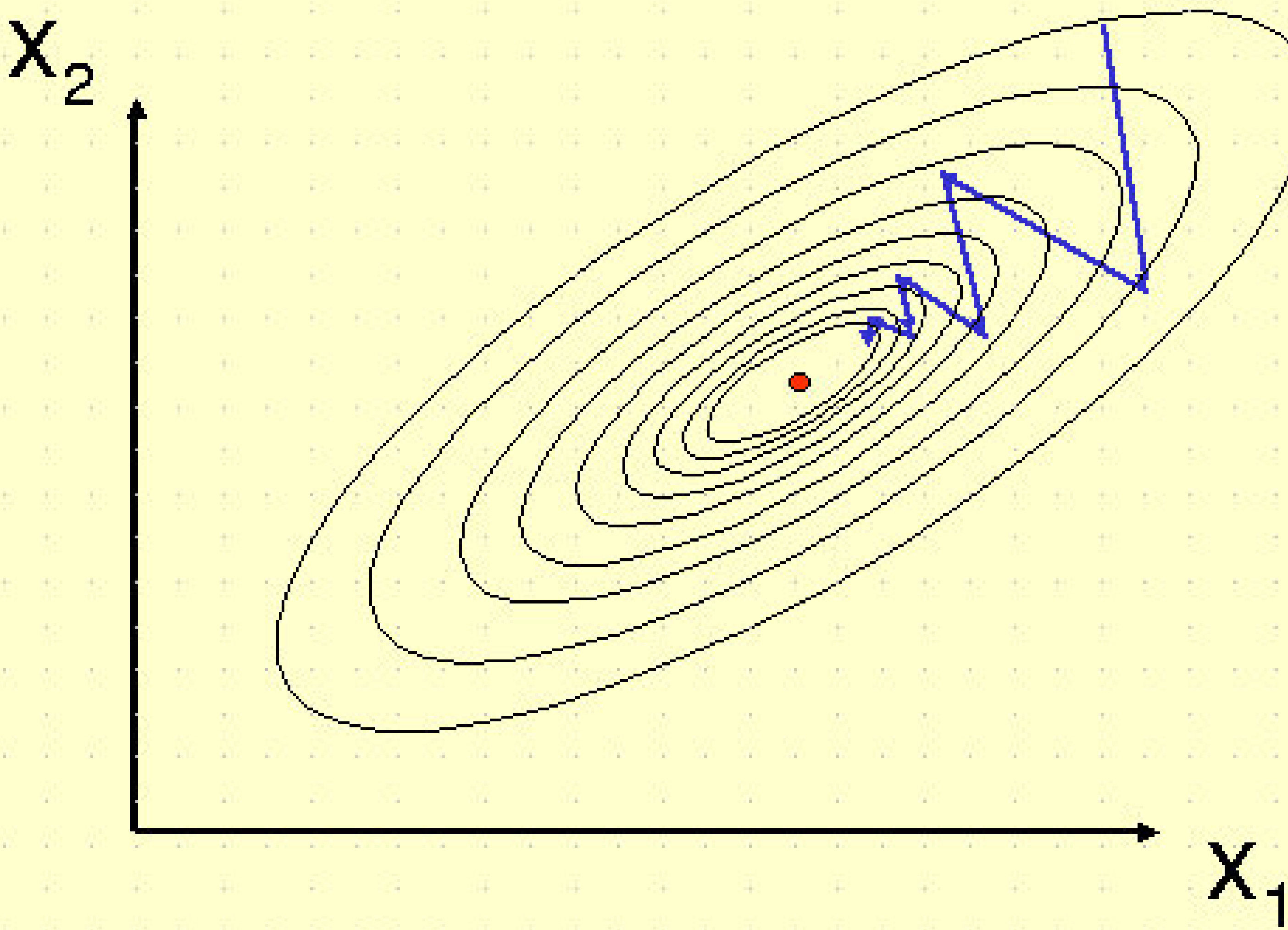


For a banana shaped surface, the minimum is much harder to find.

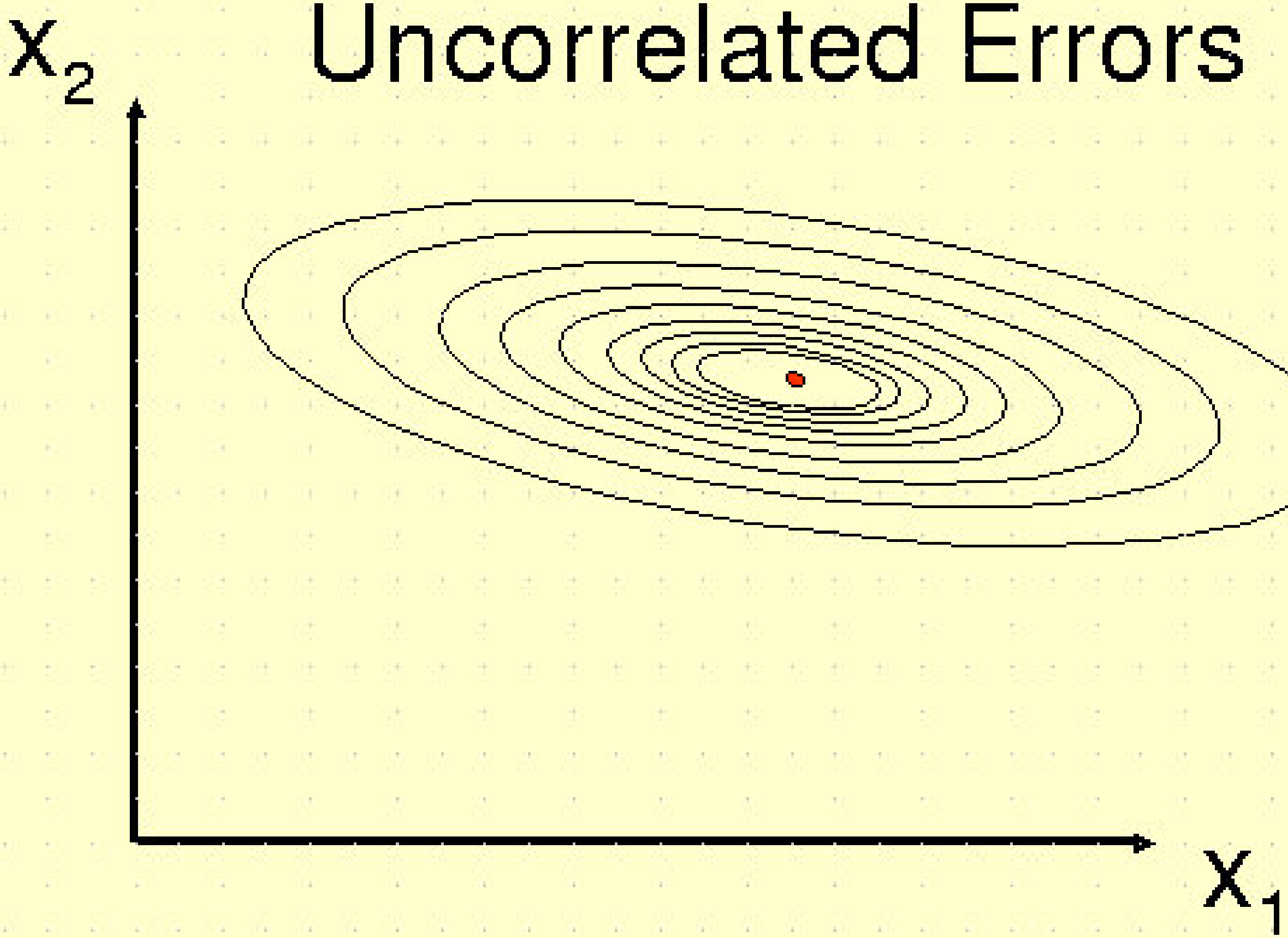


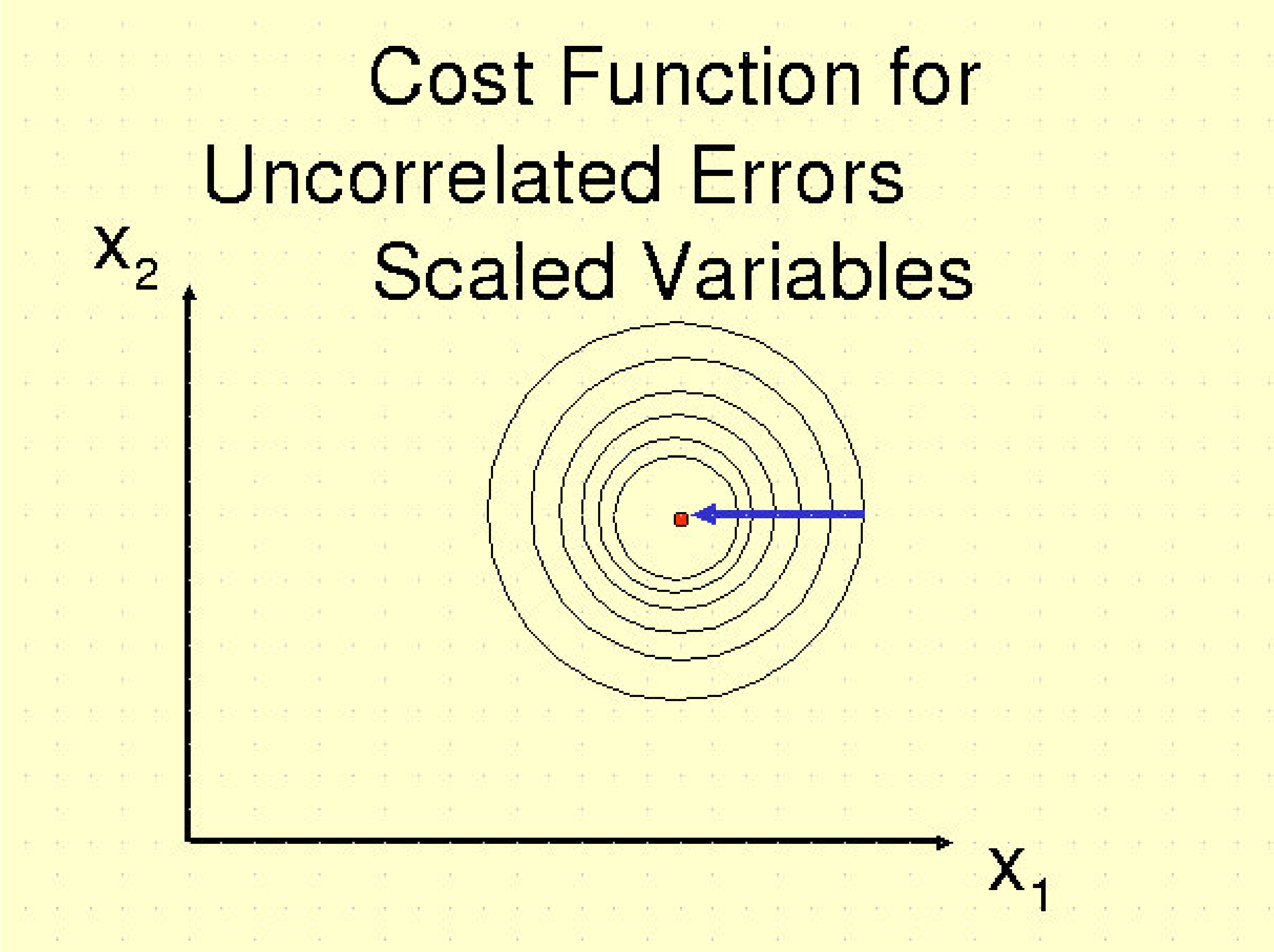
The "narrow-valley" effect. The minimization can spend many iterations zigzagging towards the minimum.

Cost Function for Correlated Errors



Cost Function for Uncorrelated Errors




Cost Function for
Uncorrelated Errors
Scaled Variables
x2
x1

Equivalence between OI and 3D-Var

We have to show that the weight matrix that multiplies the innovation $\{\mathbf{y}_o - H(\mathbf{x}_b)\} = \delta\mathbf{y}_o$ is the same as the weight matrix obtained with OI:

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However, because the methods of solution are different, their results are different, and many centers have now adopted the 3D-Var approach.

Straightforward Demonstration of Equivalence

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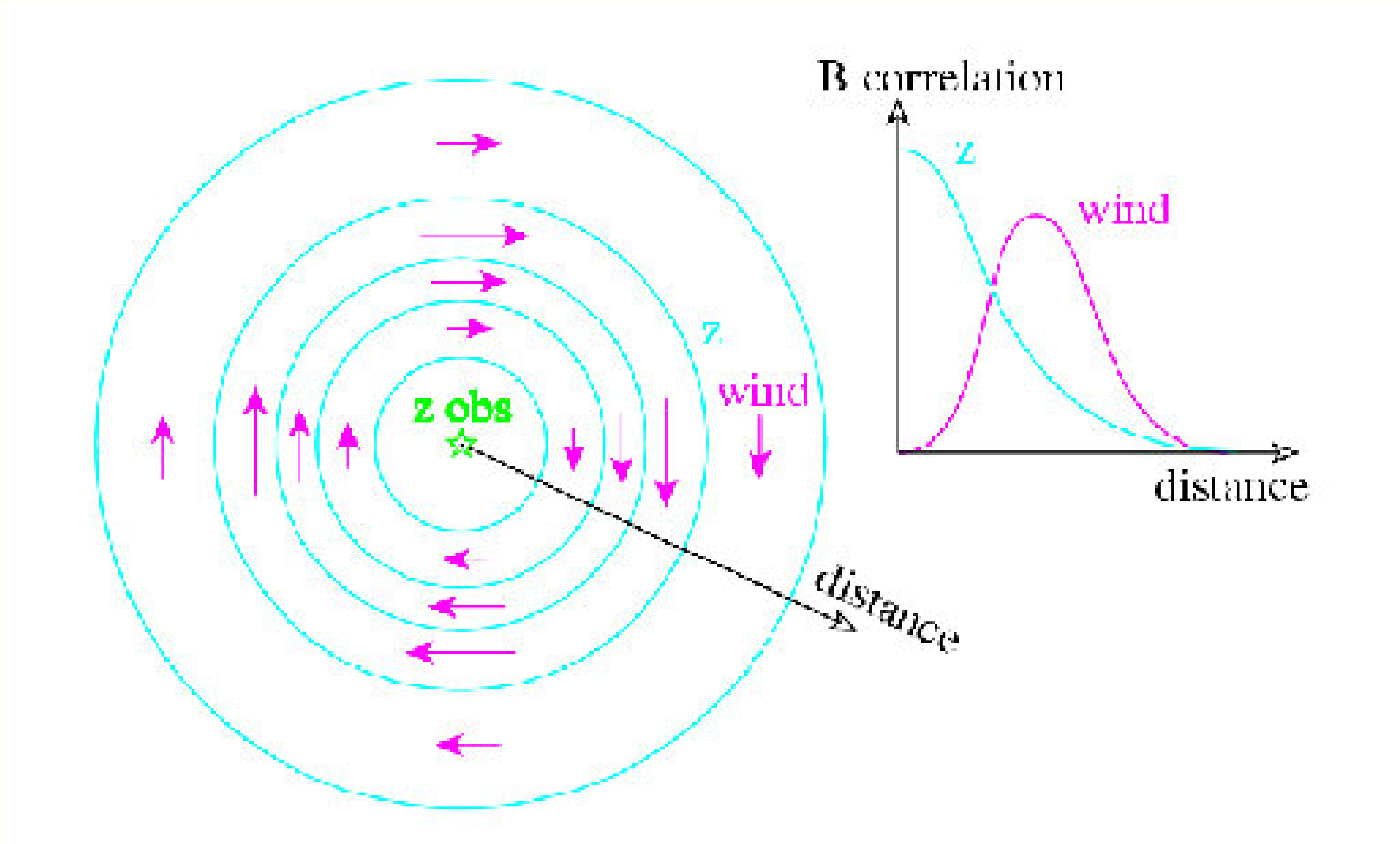
Thus 3D-Var and OI are *mathematically* equivalent.

J_B: The Conventional Method



Typical "structure function" used in OI. The autocorelation of height is an isotropic Gaussian function. By geostrophy, the cross correlation with the tangential wind is maximum where the radial gradient of the height correlation is maximum.

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This method has been shown to produce better results than previous estimates computed from *forecast minus observation* estimates.

Comparison of 3D-Var and OI

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- **The quality control of the observations becomes easier and more reliable when it is made in the space of the observations than in the space of the retrievals.**

End of §5.5