Approximations in Practical OI

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The optimal weight matrix W that minimizes the analysis error covariance is given by

$$\mathbf{W} = \mathbf{B}\mathbf{H}^T(\mathbf{H}\mathbf{B}\mathbf{H}^T + \mathbf{R})^{-1}$$

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Some scientists argue that the name statistical interpolation should be used instead of optimal interpolation. But the latter is generally used.

OI in Physical Space

OI is typically performed in physical space, either grid point by grid point or over limited volumes.

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To make the formulation in physical space clearer, we expand the matrix equations:

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\mathbf{B} = \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix} & \mathbf{H} = \begin{bmatrix} h_{11} & \dots & h_{1n} \\ \vdots & & \vdots \\ h_{p1} & \dots & h_{pn} \end{bmatrix} \\[2em]
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Note the orders of these matrices.

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For a single variable, there are n grid points.

If we are considering several variables, n is the product of the number of grid points and the variables.

Simple Low-order Example

Consider again the OI equations in matrix form:

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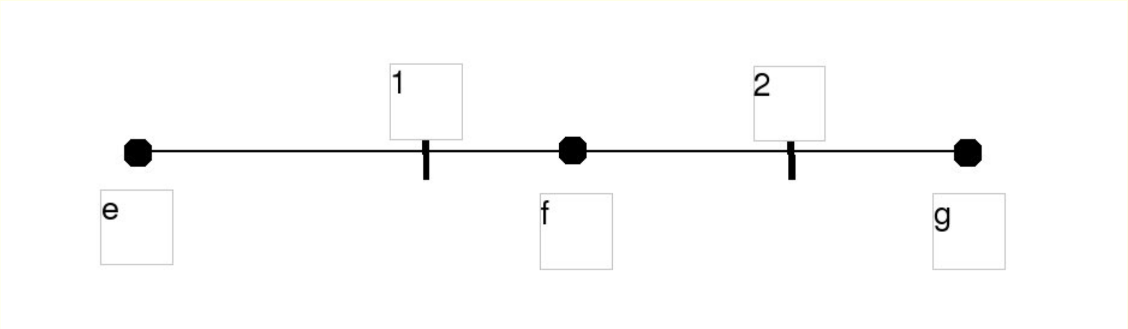

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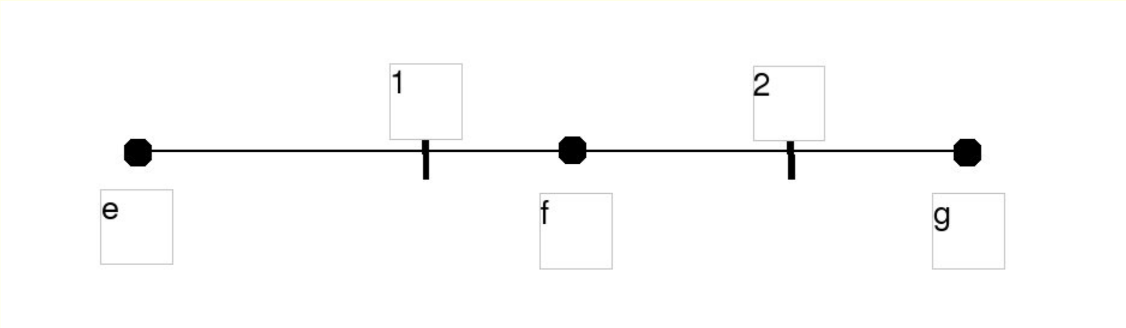

Simple example: three grid points and two observation points.

Then $\mathbf{x}^a = (x_e^a, x_f^a, x_g^a)^T$ and $\mathbf{x}^b = (x_e^b, x_f^b, x_g^b)^T$.

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For example, if we used linear interpolation, H would be

$$\mathbf{H} = \begin{pmatrix} \frac{x_f - x_1}{x_f - x_e} & \frac{x_1 - x_e}{x_f - x_e} & 0 \\ \\ 0 & \frac{x_g - x_2}{x_g - x_f} & \frac{x_2 - x_f}{x_g - x_f} \end{pmatrix}$$

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Check: Verify that the correct answer is given when an observation is located at a grid point.

The background error covariance matrix elements are the covariances between grid points:

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Note that R includes not only the instrument error, but also the representativity error.

We can now write the OI equation for a particular (single) grid point g influenced by p observations as:

$$x_g^a = x_g^b + \sum_{j=1}^{p} w_{gj} \delta y_j$$

This is the grid-point version of the vector equation

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There are equations like this for each grid point and, in multivariate analysis, for each variable at each point.

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Recall that the computation required to solve a linear system of order N typically scales as the cube of N.

SCM *versus* OI

In SCM, the weights of the observational increments depend only on their distance to the grid point.

Therefore, all observations will be given similar weight, even if a number of them are "bunched up" in one region.

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Thus, isolated observations have more independent information than observations close together; OI allows for this.

Ill-conditioned matrices

When several observations are too close together, then the solution of

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In those cases, we can compute a super-observation, combining the close individual observations. This removes the ill-posedness.

The super-observation is given a weight that takes into account the relative errors of the original observations.

Conclusion of the foregoing

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To implement OI, we need to estimate the error covariances, B and R and the observation operator H.

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The observational error covariance R is obtained from instrument error estimates.

If the measurements are independent, the matrix R is diagonal, which is a major advantage.

Background Error Covariance B

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In OI it is common to normalize the background error covariance with D, the diagonal matrix of the variances:

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Here

$$\mu_{ij} = b_{ij}/(\sqrt{b_{ii}}\sqrt{b_{jj}}) = b_{ij}/(\sigma_i \sigma_j)$$

are the correlations of the background errors at two observational points i, j, and σ_i^2 are the error variances.

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We will assume that the background error correlation between two points in the same horizontal surface is homogeneous and isotropic.

Then the background error correlation of the geopotential height depends only on the distance between the two points.

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Here $r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2$ **is the square of the distance between two points** i **and** j**, and** L_Φ**, typically of the order of 500 km, defines the background error correlation scale.**

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Gaussian functions can also be used for the vertical correlation functions:

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$$[\mu_{ij}(r_{ij})]_{\mathrm{H}} = e^{-r_{ij}^2/2L_\Phi^2}$$

Here $r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2$ is the square of the distance between two points i and j, and L_Φ, typically of the order of 500 km, defines the background error correlation scale.

Gaussian functions can also be used for the vertical correlation functions:

$$[\mu_{ij}(z)]_{\mathrm{V}} = e^{-z^2/2L_z^2}$$

Then the total correlation is the product of horizontal and vertical:

$$\mu_{ij} = [\mu_{ij}(r_{ij})]_{\mathrm{H}} \times [\mu_{ij}(z)]_{\mathrm{V}}$$

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For example, the background error correlation between δu and δv is:

$$E(\delta u_i \delta v_j) = -\frac{g}{f_i}\frac{g}{f_j}E\left(\frac{\partial(\delta z_i)}{\partial y_i}\frac{\partial(\delta z_j)}{\partial x_j}\right)$$

Since the geopotential error at the point x_j is independent of y_i, we can combine the derivatives and write

$$E(\delta u_i \delta v_j) = -\frac{g}{f_i}\frac{g}{f_j}\frac{\partial^2 E(\delta z_i \delta z_j)}{\partial y_i \partial x_j} = -\frac{g}{f_i}\frac{g}{f_j}\frac{\partial^2 b_{ij}}{\partial y_i \partial x_j} = -\frac{g^2}{f_i}\frac{\sigma_z^2}{f_j}\frac{\partial^2 \mu_{ij}}{\partial y_i \partial x_j}$$

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The standard deviation of the wind increments can also be derived from the geostrophic relationship [*]:

$$\sigma_u = E(\delta u_i^2)^{1/2} = (g\sigma_z/f_i)\,, \qquad \sigma_v = E(\delta v_j^2)^{1/2} = (g\sigma_z/f_j)$$

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So, we obtain the correlation of the increments of the two wind components by dividing by these standard deviations:

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So, we obtain the correlation of the increments of the two wind components by dividing by these standard deviations:

$$\rho_{u,v} = -\partial^2 \mu_{ij}/\partial y_i \partial x_j\,.$$

[*] Detail to be clarified later.

Similarly, we can obtain the correlations between the increments of any two of the variables at points i and j:

$$\rho_{h,h} = \mu_{ij}\,, \quad \rho_{h,u} = -\frac{\partial \mu_{ij}}{\partial y_i}\,, \quad \rho_{u,h} = -\frac{\partial \mu_{ij}}{\partial y_j}\,.$$

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Exercise: **Assume the height correlation function is Gaussian:**

$$\mu_{ij} = e^{-r_{ij}^2/2L_\phi^2}$$

where $r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2$**.**

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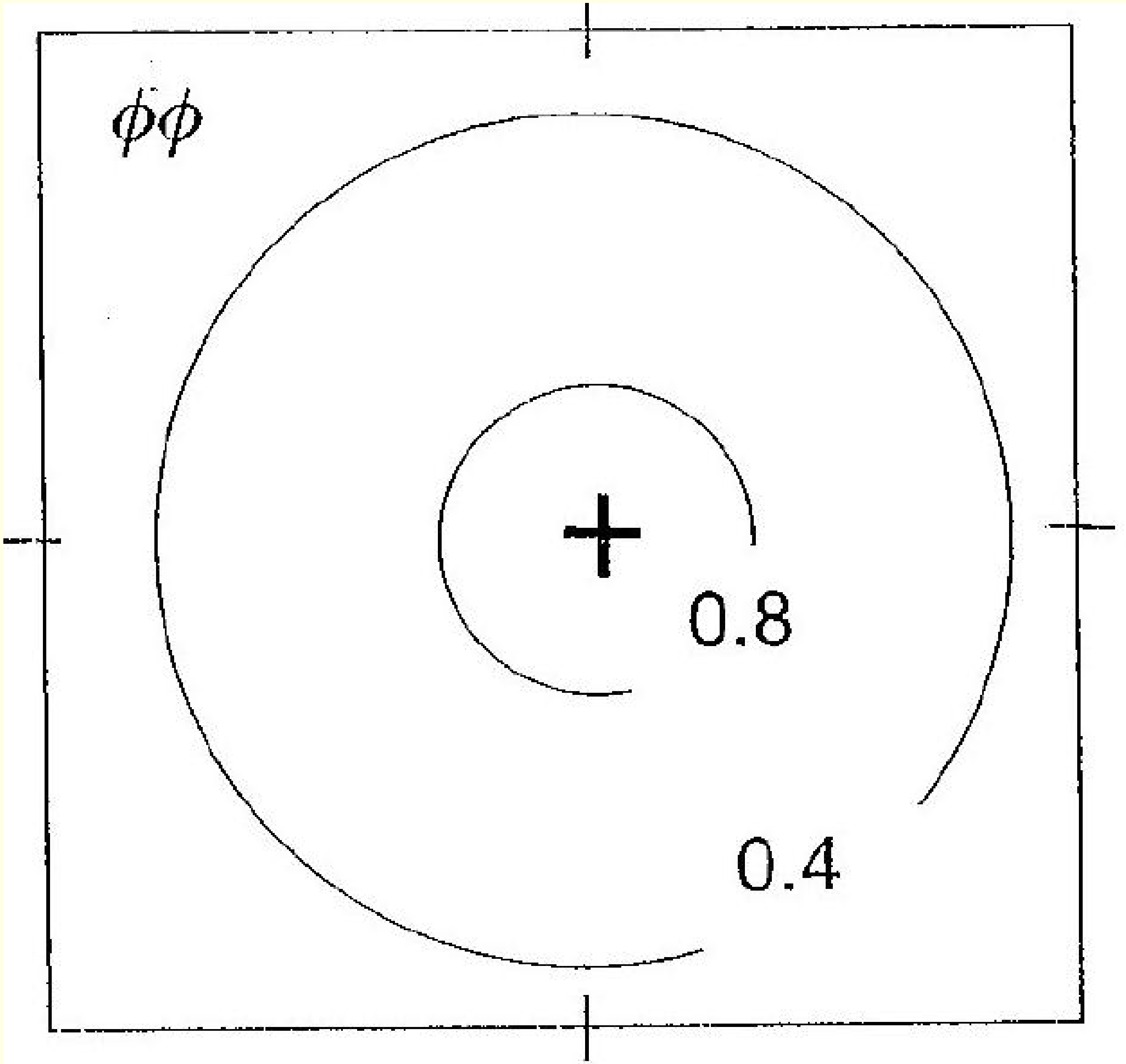
Derive the expressions for the other correlations.

$\star \quad \star \quad \star$

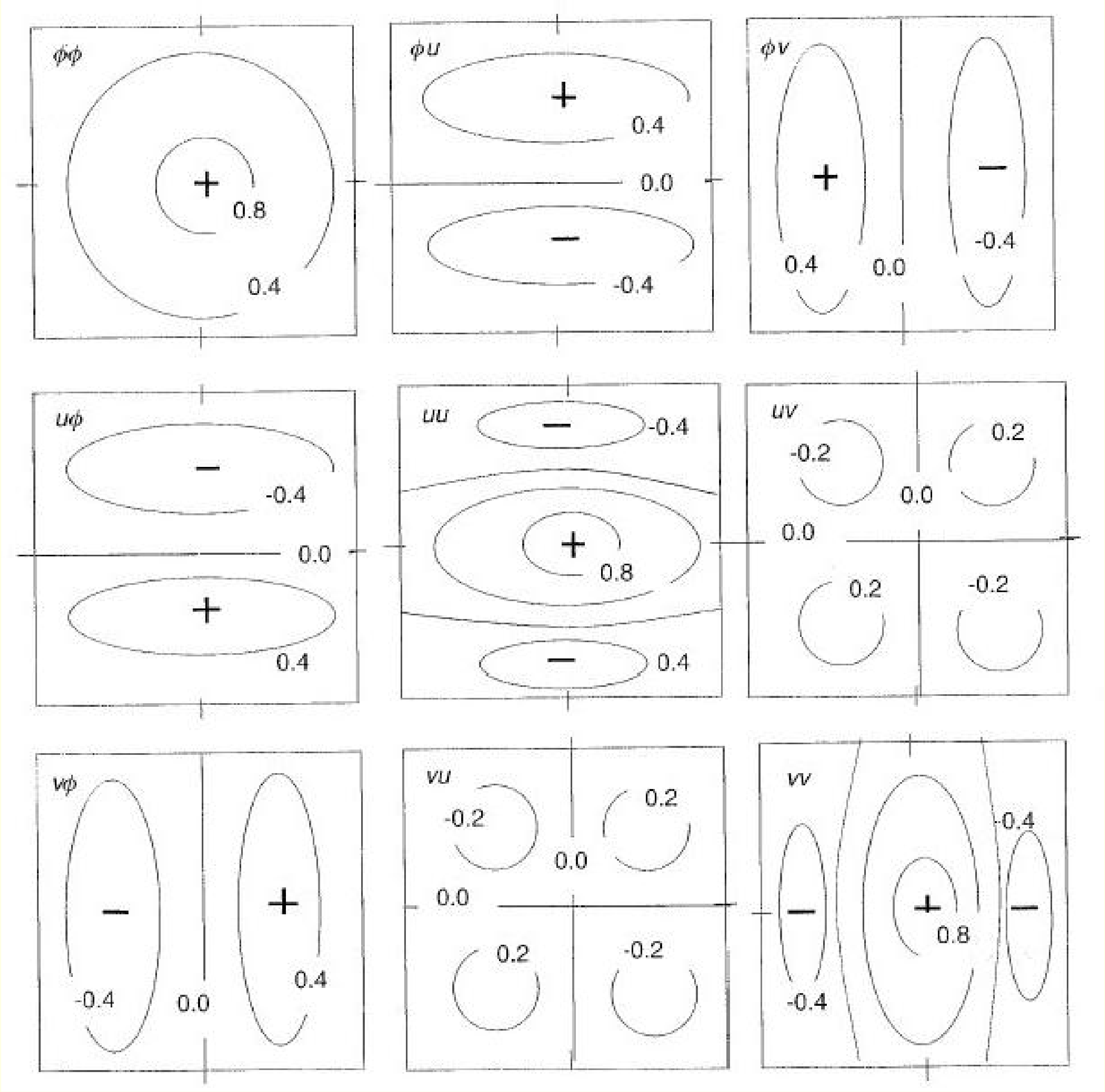
The following figure shows schematically the shape of of the correlation function for geopotential height used in OI.



Schematic illustration of the correlation of Φ-Φ.

The following figure shows schematically the shape of typical wind/height correlation functions used in OI.

Note that the u–h correlations have the opposite sign than the h–u correlations because the first and second variables correspond to the first and second points i and j respectively.



Correlation and cross-correlation functions.

Other Practical Limitations

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In addition, it is common to select the observations to be included in solving the linear system for the weight coefficients, depending on the computer resources available for the analysis, allowing for a maximum number of observations affecting each grid point.

Rules for the selection of the subset of observations to be used typically depend on the distance to the grid point (within a maximum radius of influence), the types of observations (giving priority to the most accurate) and their distribution.

End of §5.4