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If the amplitude of the motion is small, the horizontal structure is then governed by a system equivalent to the linear shallow water equations.

These equations were first derived by Laplace in his discussion of tides in the atmosphere and ocean.

They are called the Laplace Tidal Equations.

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The basic state variables satisfy the gas law, and are in hydrostatic balance:

$$\bar{p} = \mathcal{R}\bar{\rho}T_0 \qquad \textbf{and} \qquad \frac{d\bar{p}}{dz} = -g\bar{\rho}$$

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The variations of mean pressure and density follow:

$$\bar{p}(z) = p_0 \exp(-z/H) \, , \qquad \bar{\rho}(z) = \rho_0 \exp(-z/H) \, ,$$

where $H = p_0/g\rho_0 = \mathcal{R}T_0/g$ is the atmospheric scale-height.

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Exercise: Confirm this.

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The horizontal momentum, continuity and thermodynamic equations become

$$\begin{aligned}
\frac{\partial \bar{\rho} u}{\partial t} - f\bar{\rho} v + \frac{\partial p}{\partial x} &= 0 \\
\frac{\partial \bar{\rho} v}{\partial t} + f\bar{\rho} u + \frac{\partial p}{\partial y} &= 0 \\
\frac{\partial \rho}{\partial t} + \nabla\cdot\bar{\rho}\mathbf{V} &= 0 \\
\frac{1}{\gamma\bar{p}}\frac{\partial p}{\partial t} - \frac{1}{\bar{\rho}}\frac{\partial \rho}{\partial t} &= 0
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Density can be eliminated from the continuity equation by means of the thermodynamic equation.

We then get three equations for u, v and p.

We now assume that the horizontal and vertical dependencies of the perturbation quantities are separable:

$$\begin{Bmatrix} \bar{\rho}u \\ \bar{\rho}v \\ p \end{Bmatrix} = \begin{Bmatrix} U(x,y,t) \\ V(x,y,t) \\ P(x,y,t) \end{Bmatrix} Z(z) \, .$$

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$$\begin{aligned} \frac{\partial U}{\partial t} - fV + \frac{\partial P}{\partial x} &= 0 \\ \frac{\partial V}{\partial t} + fU + \frac{\partial P}{\partial y} &= 0 \\ \frac{\partial P}{\partial t} + (gh)\nabla \cdot \mathbf{V} &= 0 \end{aligned}$$

where $\mathbf{V} = (U, V)$ **is the momentum and** $h = \gamma H = \gamma \mathcal{R} T_0 / g$**.**

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This is a set of three equations for U, V, and P.

They are mathematically isomorphic to the Laplace Tidal Equations with a mean depth h (called the equivalent depth).

The Vertical Structure Equation

The vertical structure follows from the hydrostatic equation, together with the relationship $p = (\gamma g H)\rho$ implied by the thermodynamic equation.

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Solutions with more general vertical structures, and with non-vanishing vertical velocity, may be derived.

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By differentiating the momentum equations, we get equations for the vorticity and divergence tendencies, e.g.,

$$\frac{\partial\zeta}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right)$$

The vorticity, divergence and continuity equations are

$$\begin{aligned}
\frac{\partial \zeta}{\partial t} + f\delta + \beta v &= 0 \\
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This system is equivalent to the Laplace Tidal Equations. No additional approximations have been made . . .

. . . however, the vorticity and divergence forms enable us to examine various simple approximate solutions.

Mathematical Interlude

The eignefunctions of the Laplacian operator on the sphere are called spherical harmonics:

$$Y_n^m(\lambda, \phi) = \exp(im\lambda) P_n^m(\phi)$$

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The 'beta-term' in the vorticity equation may be written

$$\beta v = \frac{2\Omega\cos\phi}{a}\left(\frac{1}{a\cos\phi}\frac{\partial\psi}{\partial\lambda} + \frac{1}{a}\frac{\partial\chi}{\partial\lambda}\right)$$

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For quasi-non-divergent flow ($|\delta| \ll |\zeta|$) it becomes

$$\beta v \approx \frac{2\Omega}{a^2} \frac{\partial \psi}{\partial \lambda}$$

Rossby-Haurwitz Modes

If we suppose that the solution is quasi-nondivergent (that is, $|\delta| \ll |\zeta|$), the wind is given approximately in terms of the stream function $(u, v) \approx (-\psi_y, \psi_x)$.

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Assuming the stream function has the wave-like structure of a spherical harmonic, we substitute the expression

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We can immediately deduce an expression for the frequency:

$$\nu = \nu_R \equiv -\frac{2\Omega m}{n(n+1)}\,.$$

Repeat

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The Rossby or Rossby-Haurwitz waves are, to the first approximation, non-divergent waves which travel westward, the phase speed being greatest for the waves of largest scale.

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This also follows immediately from the assumption that the wind is both non-divergent and geostrophic:

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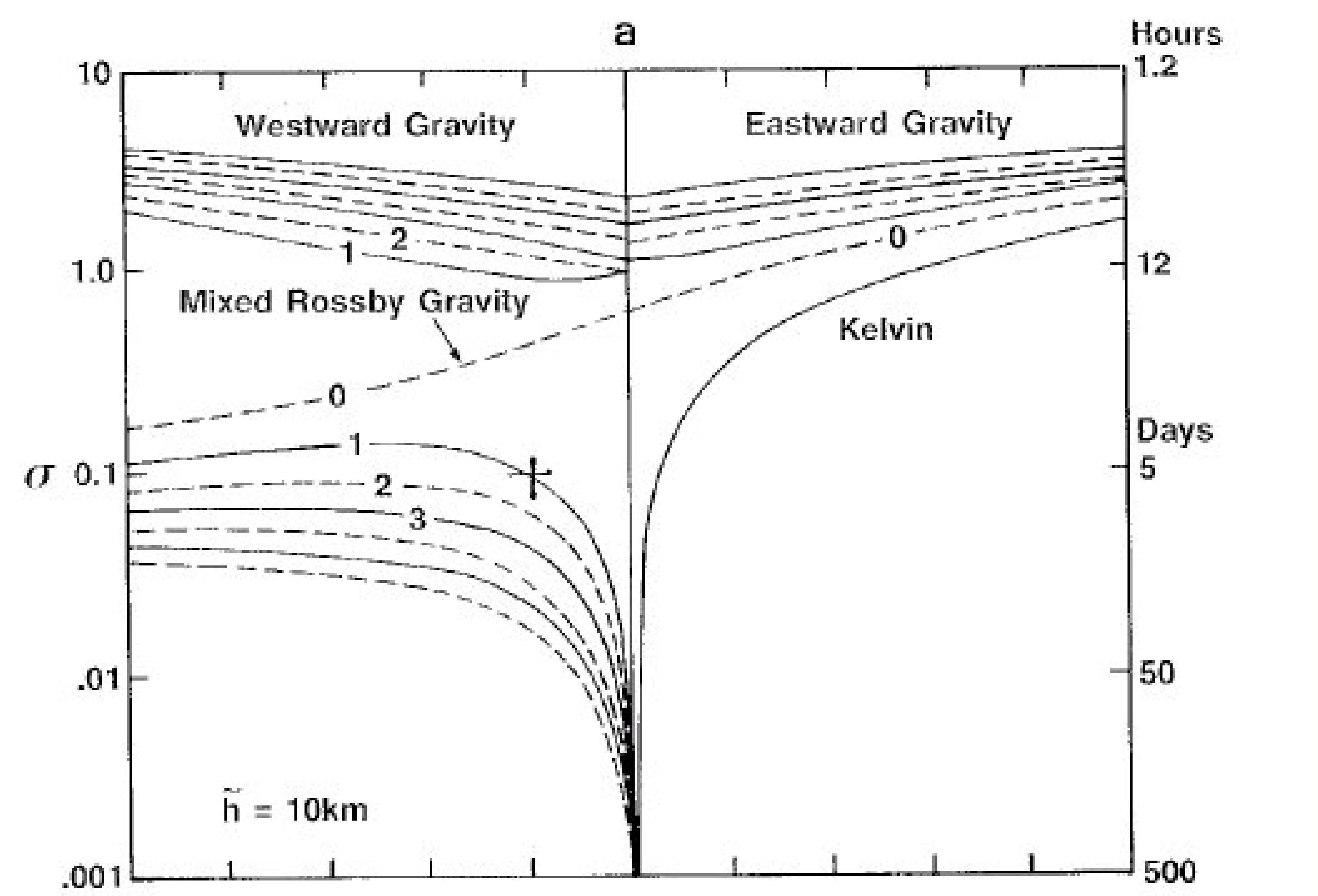
If variations of f are ignored, we can assume $P = f\psi$. The wind and pressure are in approximate geostrophic balance for Rossby-Haurwitz waves.



The eigenmodes of the Laplace TIdal Equations ($h = 10\,\mathbf{km}$).

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Then we find that

$$\nu^2 + \left(-\frac{2\Omega m}{n(n+1)} \right) \nu - \frac{n(n+1)gh}{a^2} = 0 \,.$$

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The coefficient of the second term is just the Rossby-Haurwitz frequency ν_R, so that

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the frequency of pure gravity waves.

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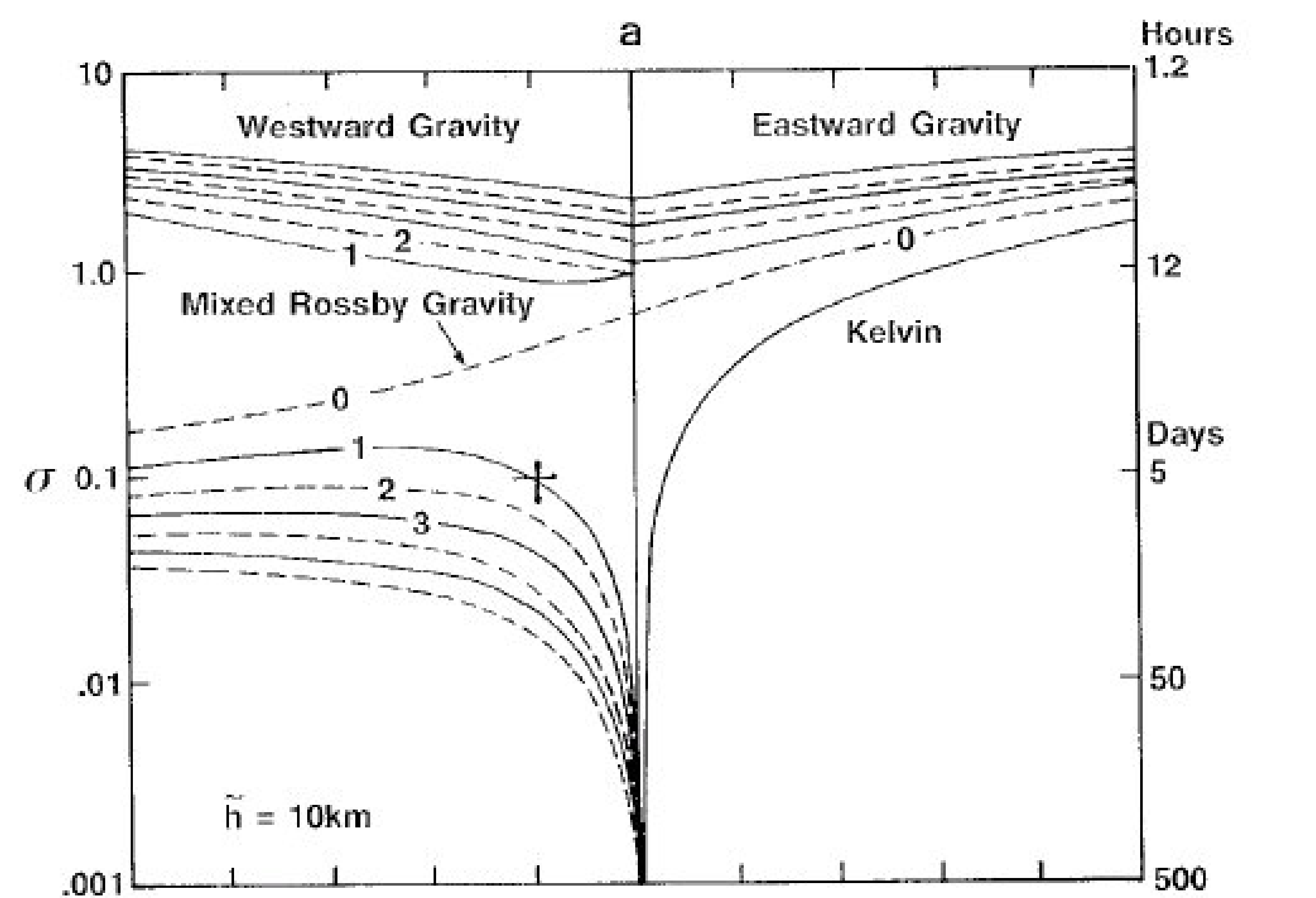
There are then two solutions, representing waves travelling eastward and westward with equal speeds.

The frequency increases approximately linearly with the total wavenumber n.



The eigenmodes of the Laplace TIdal Equations ($h = 10\,\mathbf{km}$).

Break here

Reminder on linear algebra

Let $\mathbf{M}$ be a matrix. An eigenvector $\mathbf{e}$ of $\mathbf{M}$ with eigenvalue λ satisfies

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It follows immediately that

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Normal Mode Initialization ⋆

Let X be the state vector of dependent variables.

The model equations can be written schematically as

$$\dot{\mathbf{X}} + i\mathbf{L}\mathbf{X} + \mathbf{N}(\mathbf{X}) = \mathbf{0}$$

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The vectors Y and Z are the coefficients of the LF and HF components of the flow: the slow and fast components.

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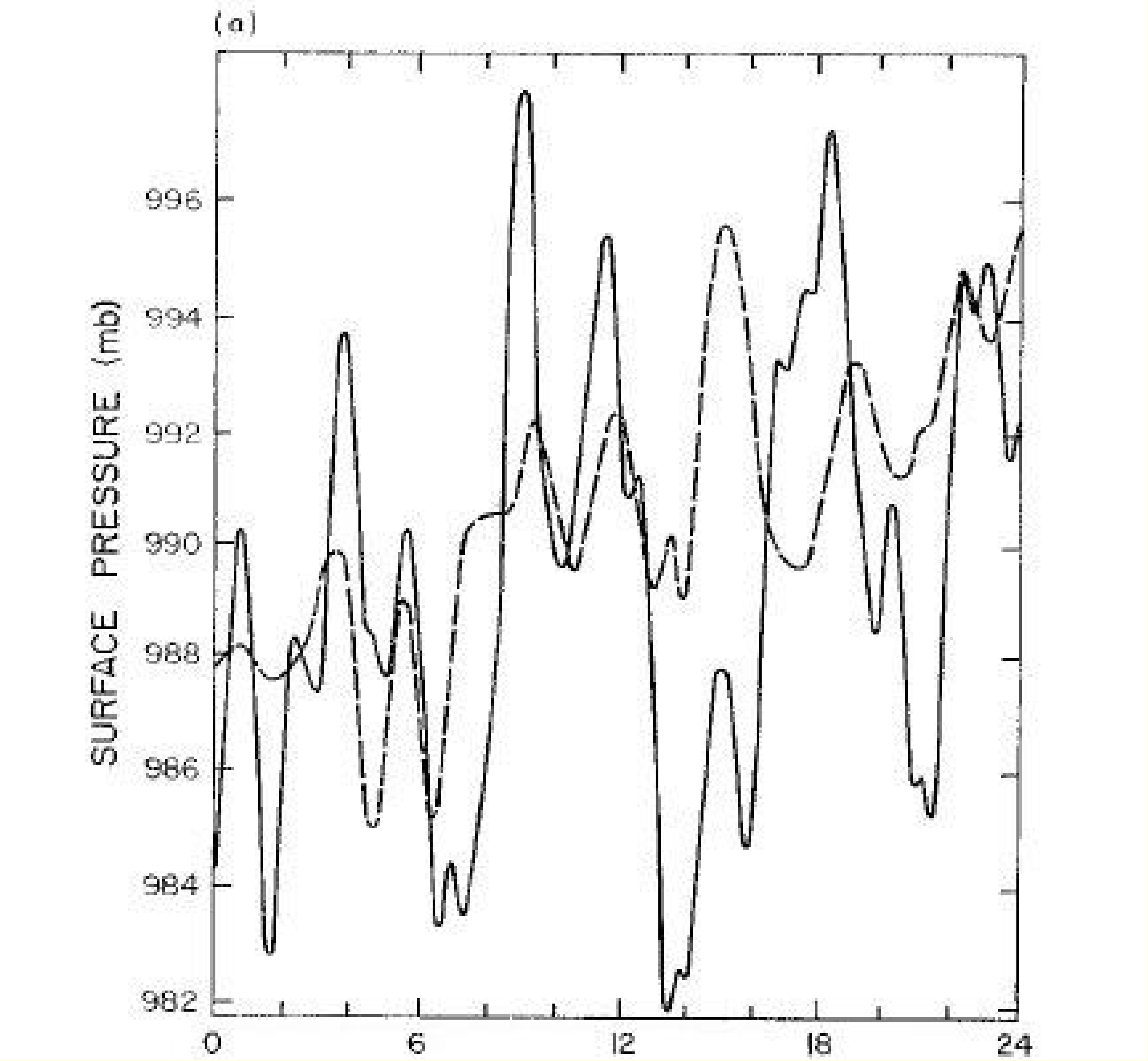
The problem of noise remains: the gravity waves are small to begin with, but they grow rapidly.



Surface pressure evolution: No Initialization and LNMI.

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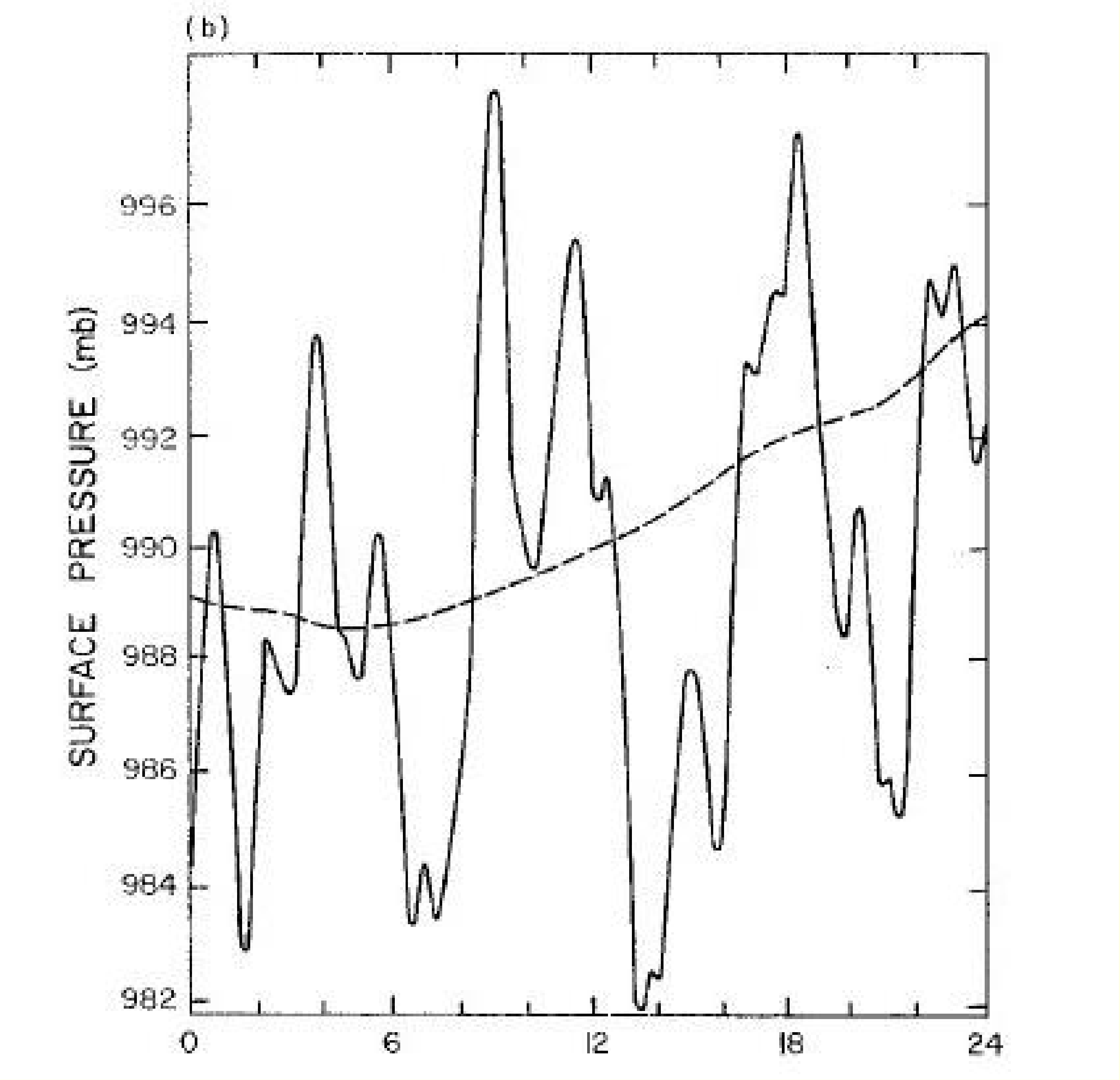
The method takes account of the nonlinear nature of the equations, and is referred to as nonlinear normal mode initialization.



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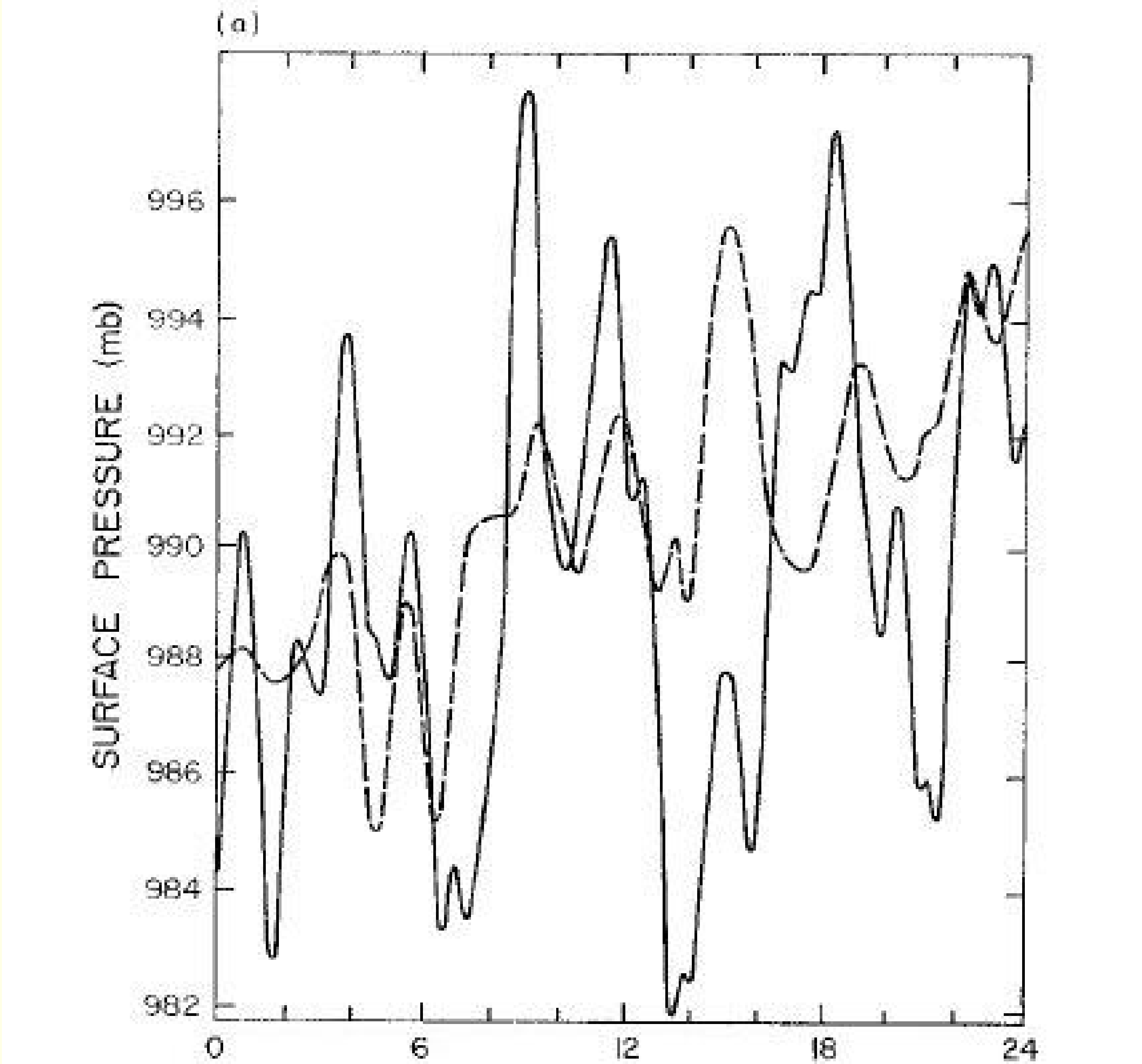

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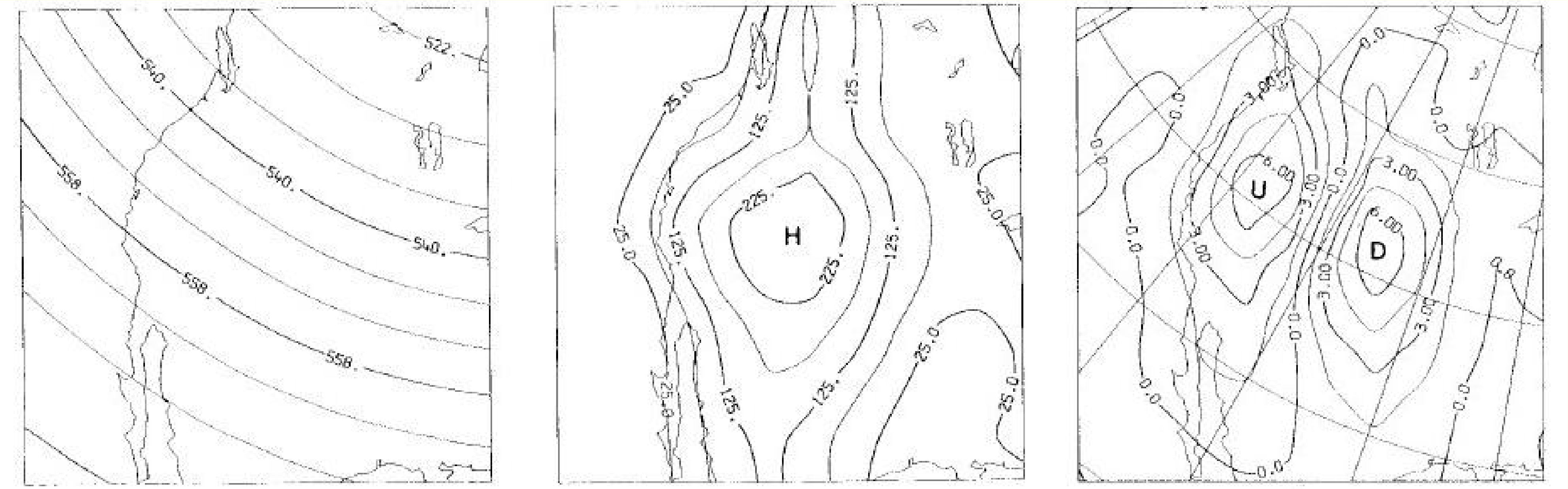

Vertical velocity for flow over the Rockies.

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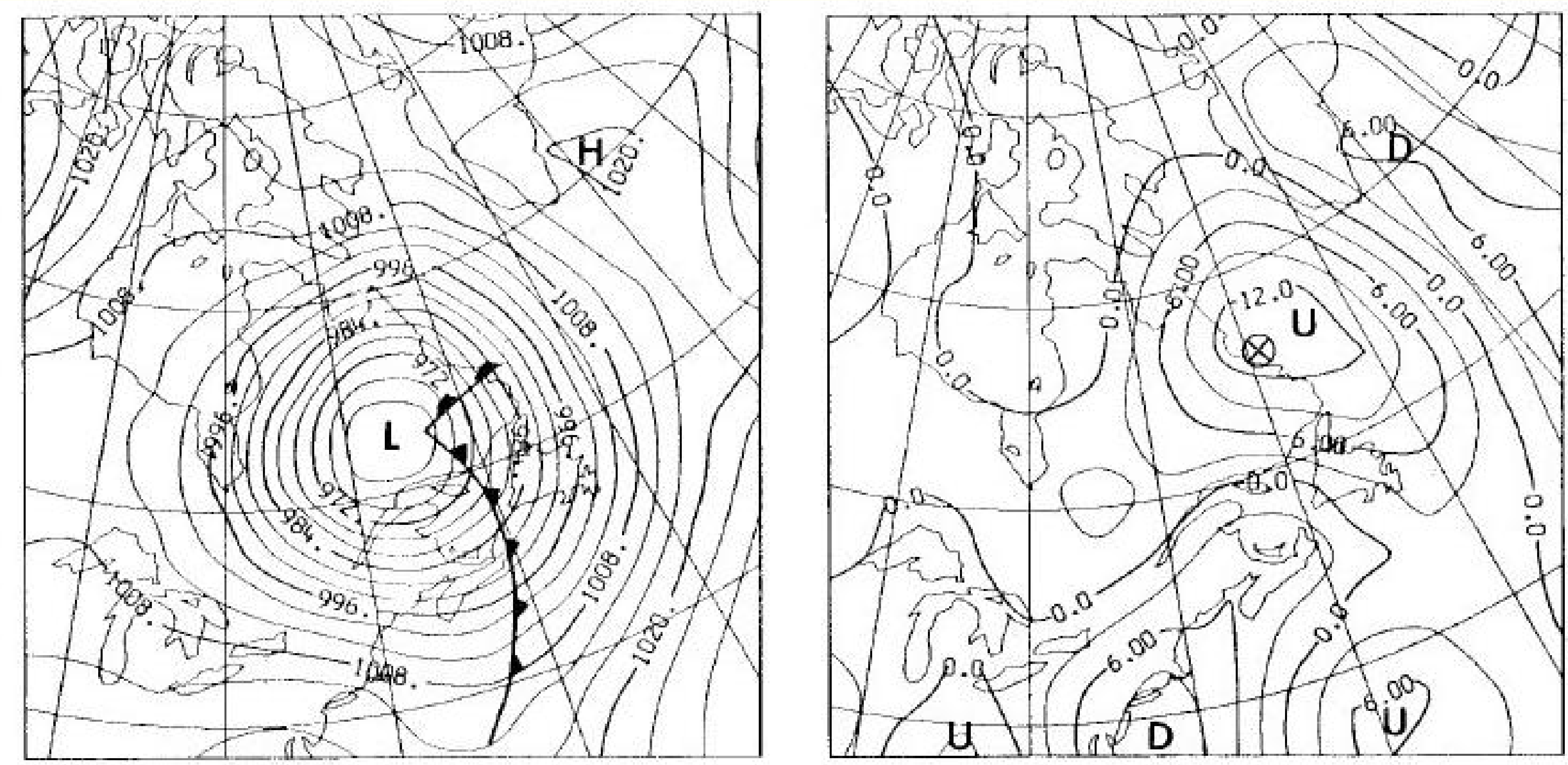

Generation of vertical velocity in frontal depression.

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Example: The Swinging Spring

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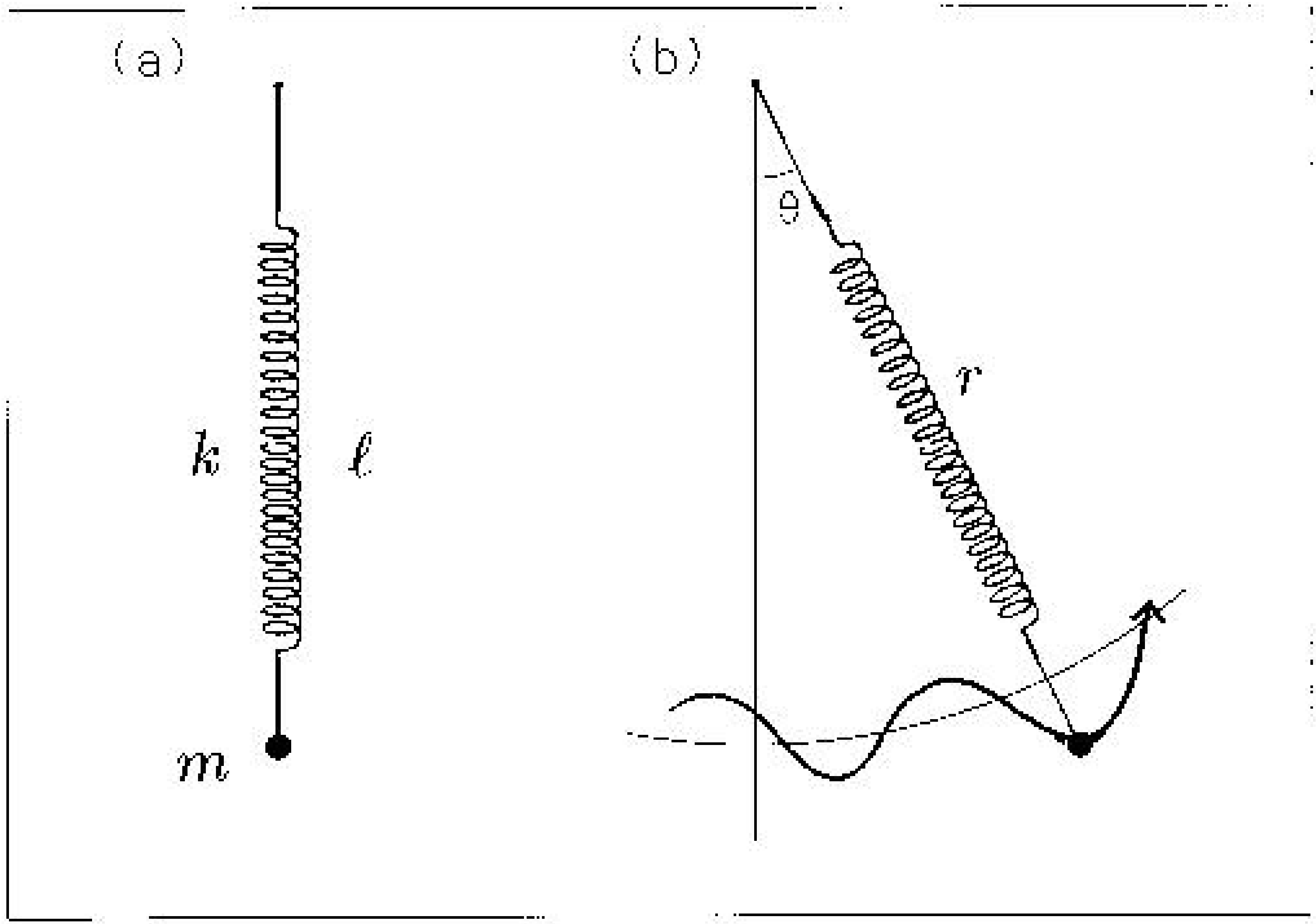
Similarly, the low frequency rotational motions are considered to correspond to the rotational or Rossby waves.



The swinging spring (2D case)

The Dynamical Equations

Let ℓ_0 be the unstretched length of the spring, k its elasticity or stiffness and m the mass of the bob.

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The Hamiltonian is (in this case) the sum of kinetic, elastic potential and gravitational potential energy:

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \tfrac{1}{2}k(r - \ell_0)^2 - mgr\cos\theta\,.$$

(If the Hamiltonian formalism is unfamiliar, the equations may be derived by considering the forces on the bob).

The (canonical) dynamical equations may now be written

$$\begin{aligned}
\dot{\theta} &= p_\theta / m r^2 \\
\dot{p}_\theta &= -m g r \sin\theta \\
\dot{r} &= p_r / m \\
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These equations may be written symbolically in vector form

$$\dot{\mathbf{X}} + \mathbf{L}\mathbf{X} + \mathbf{N}(\mathbf{X}) = 0$$

where $\mathbf{X} = (\theta, p_\theta, r, p_r)^{\mathrm{T}}$, $\mathbf{L}$ is the matrix of coefficients of the linear terms and $\mathbf{N}$ is a nonlinear vector function.

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$$\mathbf{X} = \begin{pmatrix} \mathbf{Y} \\ \mathbf{Z} \end{pmatrix} , \qquad \textbf{where} \quad \mathbf{Y} = \begin{pmatrix} \theta \\ p_\theta \end{pmatrix} \quad \textbf{and} \quad \mathbf{Z} = \begin{pmatrix} r' \\ p_r \end{pmatrix} ,$$

The linear slow and fast modes evolve independently.

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The remaining two equations yield

$$\ddot{r}' + (k/m)r' = 0\,,$$

the equations for elastic oscillations with frequency

$$\omega_{\mathrm{E}} = \sqrt{\frac{k}{m}}\,.$$

We define the ratio of the rotational and elastic frequencies:

$$\omega_{\mathrm{R}} = \sqrt{\frac{g}{\ell}} \,, \qquad \omega_{\mathrm{E}} = \sqrt{\frac{k}{m}} \,, \qquad \epsilon \equiv \left(\frac{\omega_{\mathrm{R}}}{\omega_{\mathrm{E}}} \right) \,.$$

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It is easily shown that $\epsilon < 1$, so the rotational frequency is always less than the elastic.

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We assume that the parameters are such that

$$\epsilon \ll 1$$

In this case the linear normal modes are clearly distinct:

- **The rotational mode has low frequency (LF)**
- **The elastic mode has high frequency (HF).**

Linear and Nonlinear Initialization

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They evolve without interaction.

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It is clear from the equations that linear initialization will not ensure permanent absence of HF motions . . .

. . . the nonlinear LF terms generate radial momentum.

To achieve better results, we set the initial tendency of the HF components to zero:

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For the spring, we can deduce explicit expressions for the initial conditions:

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Thus, given arbitrary initial conditions $\mathbf{X} = (\theta, p_\theta, r, p_r)^{\mathrm{T}}$,

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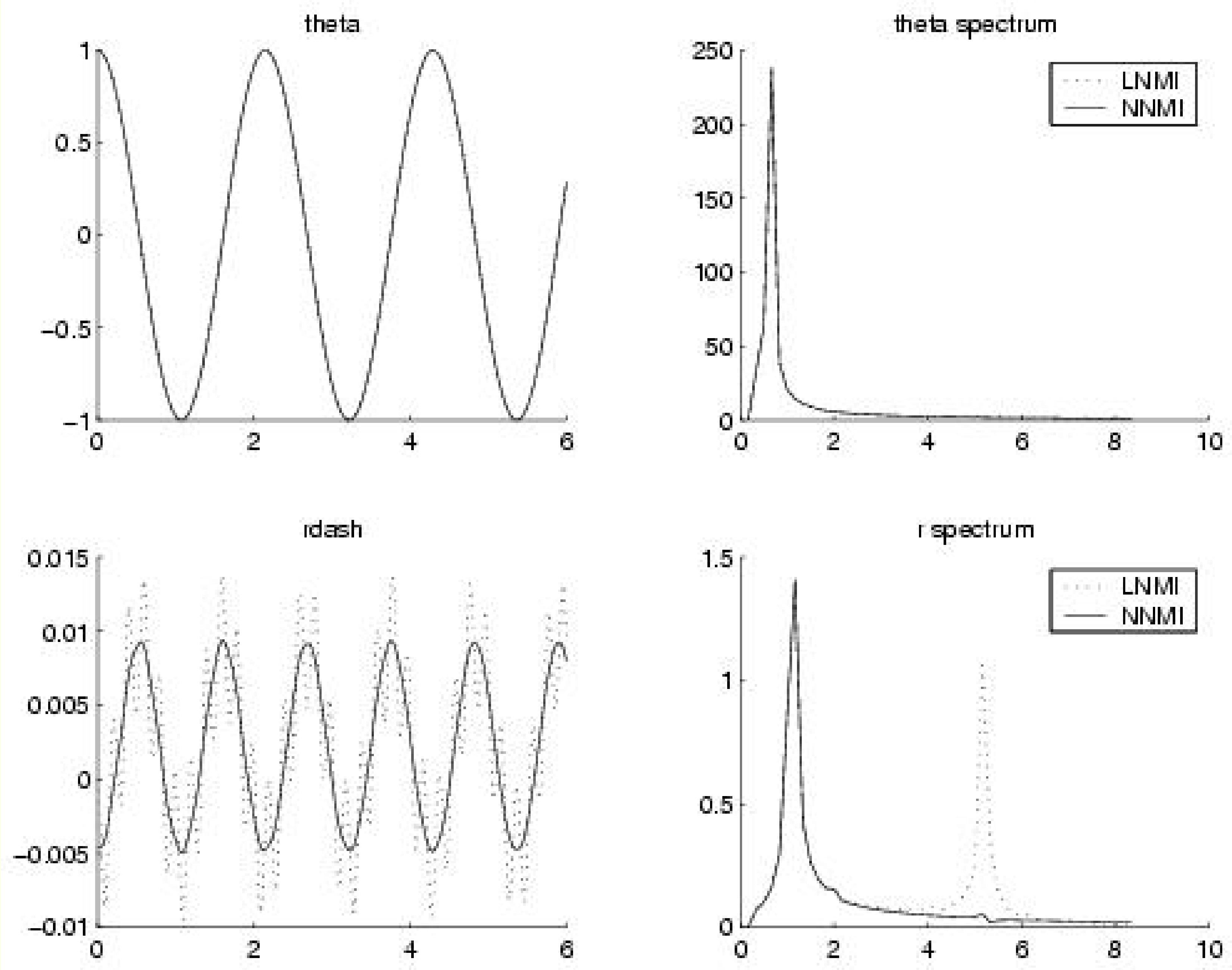
Does it work? An example to follow shows that it does!



Solution of swinging spring equations for linear (LNMI) and nonlinear (NNMI) initialization.

A Numerical Example

In the accompanying figure, we show the results of two integrations of the spring equations.

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Thus, $\epsilon = 0.1$ **and the periods of the swinging and springing motions are respectively**

$$\tau_{\mathrm{R}} = 2\,\mathrm{s} \qquad \textbf{and} \qquad \tau_{\mathrm{E}} = 0.2\,\mathrm{s}\,.$$

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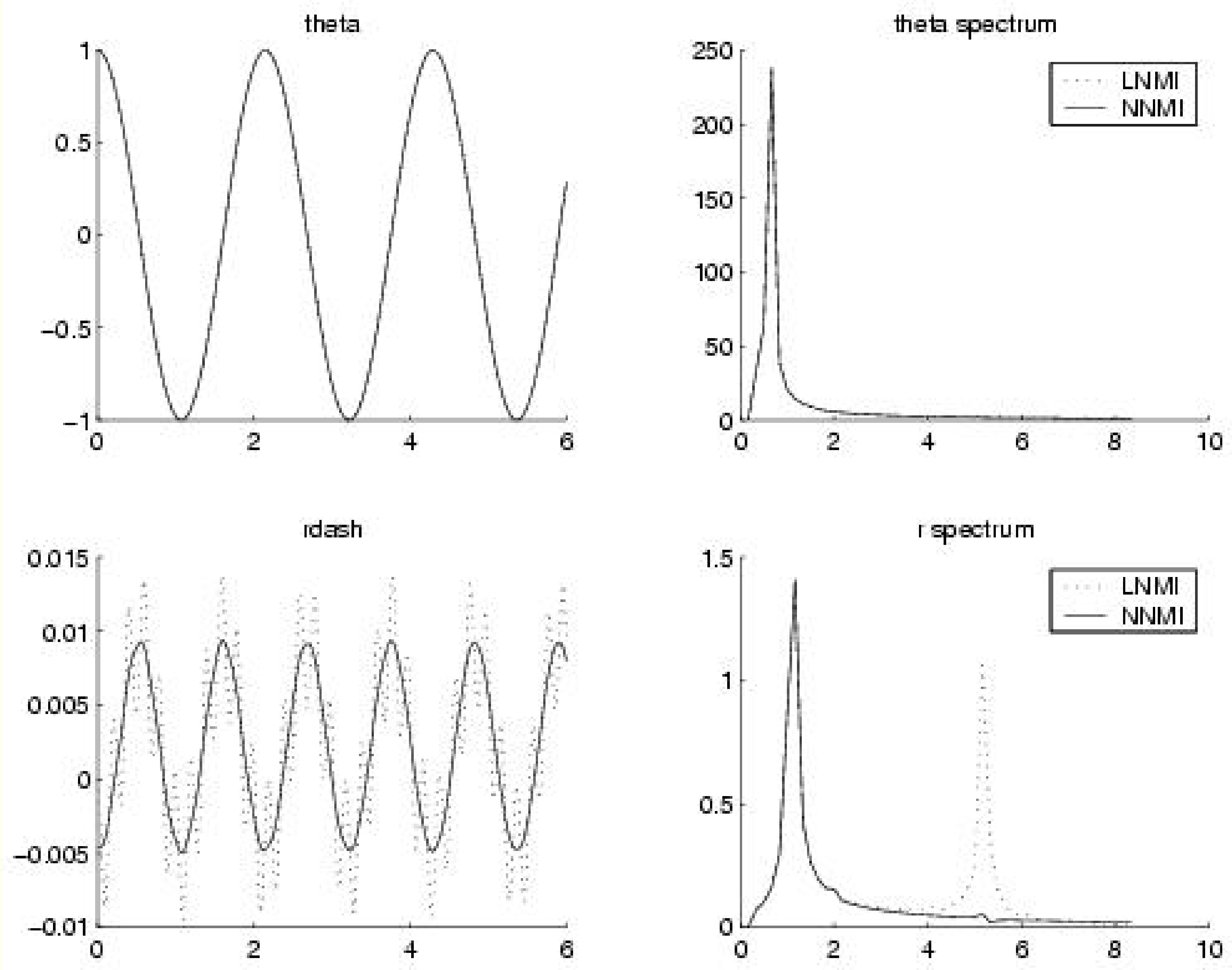
The equations are integrated over a period of 6 seconds



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The lower panels are for the fast variable r.

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When nonlinearly initialized, this peak is removed: only the peak at 1 Hz remains.

This is the 'balanced fast motion'.

The 'balanced fast motion' can be understood physically . . .

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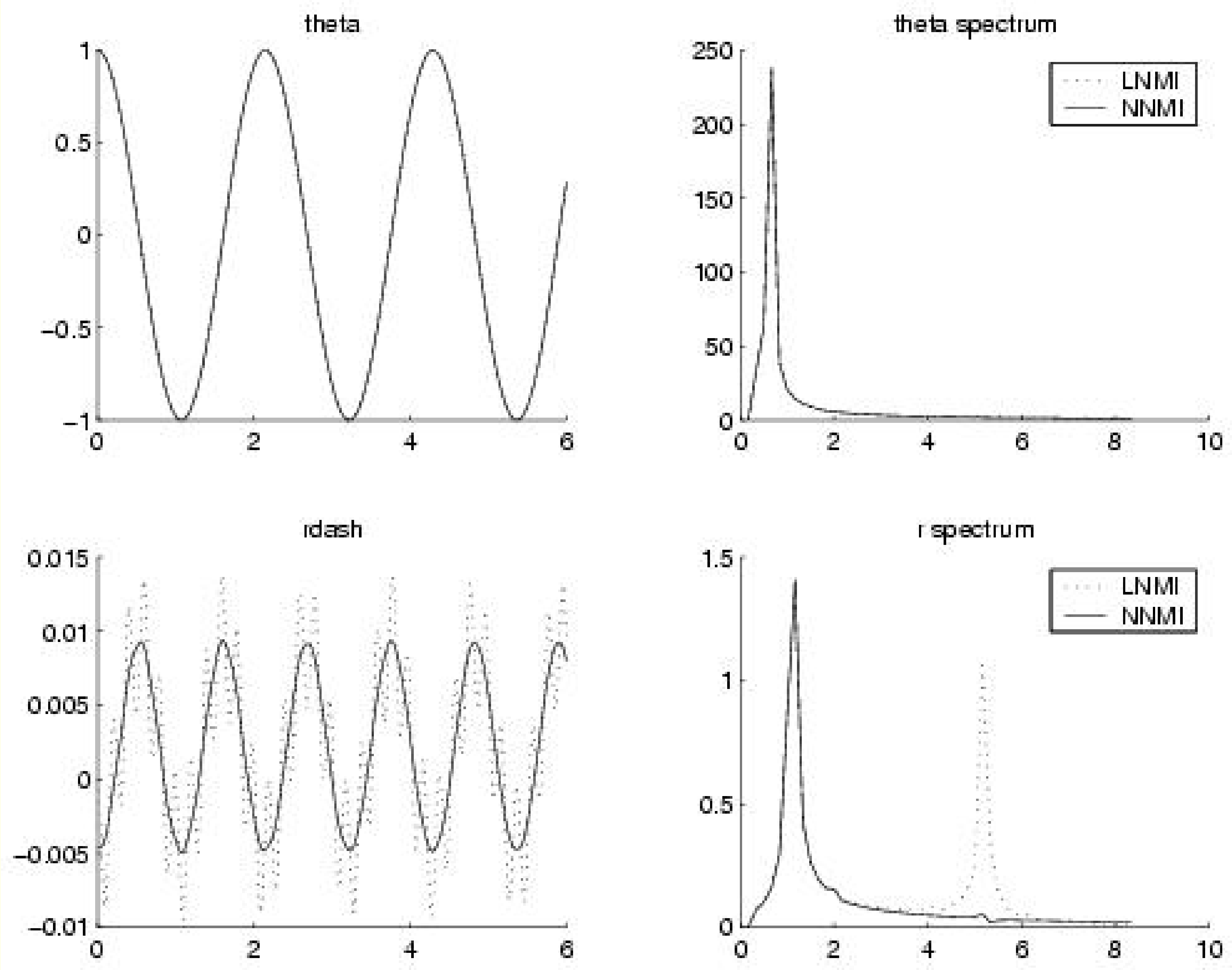
The radial variation does not disappear for balanced motion, but it is of low frequency.

The balanced fast motion is said to be 'slaved' (or, better, enslaved) to the slow motion.



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End of §4.2