§4.2. Normal Mode Initialization

We start by considering the *Normal Mode Oscillations* of the atmopshere.

The solutions of the model equations can be separated, by spectral analysis, into two sets of linear normal modes:

- Slow rotational components or **Rossby modes**
- High frequency **gravity-inertia modes**

If the amplitude of the motion is small, the horizontal structure is then governed by a system equivalent to the linear shallow water equations.

These equations were first derived by **Laplace** in his discussion of tides in the atmosphere and ocean.

They are called the **Laplace Tidal Equations**.

The Laplace Tidal Equations

The simplest means of deriving the linear shallow water equations from the primitive equations is to assume that the vertical velocity vanishes identically.

We assume that the motions can be described as small perturbations about a state of rest, with constant temperature T_0, and pressure $\bar{p}(z)$ and density $\bar{\rho}(z)$ varying with height.

The basic state variables satisfy the gas law, and are in hydrostatic balance:

$$\bar{p} = \mathcal{R}\bar{\rho}T_0 \qquad \text{and} \qquad \frac{d\bar{p}}{dz} = -g\bar{\rho}$$

The variations of mean pressure and density follow:

$$\bar{p}(z) = p_0 \exp(-z/H), \qquad \bar{\rho}(z) = \rho_0 \exp(-z/H),$$

where $H = p_0/g\rho_0 = \mathcal{R}T_0/g$ is the atmospheric scale-height.

Exercise: Confirm this.

We consider only motions for which the vertical component of velocity vanishes identically, $w \equiv 0$.

Let u, v, p and ρ denote variations about the basic state, each of these being a small quantity.

The horizontal momentum, continuity and thermodynamic equations become

$$\frac{\partial \bar{\rho}u}{\partial t} - f\bar{\rho}v + \frac{\partial p}{\partial x} = 0$$
$$\frac{\partial \bar{\rho}v}{\partial t} + f\bar{\rho}u + \frac{\partial p}{\partial y} = 0$$
$$\frac{\partial \rho}{\partial t} + \nabla\cdot\bar{\rho}\mathbf{V} = 0$$
$$\frac{1}{\gamma\bar{p}}\frac{\partial p}{\partial t} - \frac{1}{\bar{\rho}}\frac{\partial \rho}{\partial t} = 0$$

Density can be eliminated from the continuity equation by means of the thermodynamic equation.

We then get **three equations for u, v and p.**

We now assume that the horizontal and vertical dependencies of the perturbation quantities are separable:

$$\left\{\begin{matrix} \bar{\rho}u \\ \bar{\rho}v \\ p \end{matrix}\right\} = \left\{\begin{matrix} U(x,y,t) \\ V(x,y,t) \\ P(x,y,t) \end{matrix}\right\} Z(z) .$$

The momentum and continuity equations can be written

$$\frac{\partial U}{\partial t} - fV + \frac{\partial P}{\partial x} = 0$$
$$\frac{\partial V}{\partial t} + fU + \frac{\partial P}{\partial y} = 0$$
$$\frac{\partial P}{\partial t} + (gh)\nabla \cdot \mathbf{V} = 0$$

where $\mathbf{V} = (U, V)$ is the momentum and $h = \gamma H = \gamma\mathcal{R}T_0/g$.

This is a set of three equations for U, V, and P.

They are mathematically isomorphic to the **Laplace Tidal Equations** with a mean depth h (called the **equivalent depth**).

The Vertical Structure Equation

The **vertical structure** follows from the hydrostatic equation, together with the relationship $p = (\gamma g H)\rho$ implied by the thermodynamic equation.

It is determined by

$$\frac{dZ}{dz} + \frac{Z}{\gamma H} = 0 \,,$$

The solution of this is $Z = Z_0 \exp(-z/\gamma H)$, where Z_0 is the amplitude at $z = 0$.

If we set $Z_0 = 1$, then U, V and P give the momentum and pressure fields at the earth's surface.

These variables all **decay exponentially with height**.

It follows that u and v actually increase with height as $\exp(\kappa z/H)$, but the kinetic energy decays.

Solutions with more general vertical structures, and with non-vanishing vertical velocity, may be derived.

The vorticity, divergence and continuity equations are

$$\frac{\partial \zeta}{\partial t} + f\delta + \beta v = 0$$

$$\frac{\partial \delta}{\partial t} - f\zeta + \beta u + \nabla^2 P = 0$$

$$\frac{\partial P}{\partial t} + gh\delta = 0 \,.$$

This system is equivalent to the **Laplace Tidal Equations**. No additional approximations have been made ...

... however, the vorticity and divergence forms enable us to examine various <u>simple approximate solutions</u>.

Vorticity and Divergence

We examine the solutions of the Laplace Tidal Equations in some enlightening limiting cases.

By means of the **Helmholtz Theorem**, a general horizontal wind field **V** may be partitioned into **rotational and divergent components**

$$\mathbf{V} = \mathbf{V}_\psi + \mathbf{V}_\chi = \mathbf{k} \times \nabla \psi + \nabla \chi \,.$$

The stream function ψ and velocity potential χ are related to the vorticity and divergence by the **Poisson equations**

$$\nabla^2 \psi = \zeta \qquad \text{and} \qquad \nabla^2 \chi = \delta \,.$$

By differentiating the momentum equations, we get equations for the vorticity and divergence tendencies, e.g.,

$$\frac{\partial \zeta}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right)$$

Mathematical Interlude

The eignefunctions of the Laplacian operator on the sphere are called **spherical harmonics**:

$$Y_n^m(\lambda, \phi) = \exp(im\lambda)P_n^m(\phi)$$

where $P_n^m(\phi)$ are the **associated Legendre functions**.

We have

$$\nabla^2 Y_n^m = -\frac{n(n+1)}{a^2}Y_n^m \,.$$

The **zonal wavenumber** is m. The **total wavenumber** is n.

The 'beta-term' in the vorticity equation may be written

$$\beta v = \frac{2\Omega \cos\phi}{a}\left(\frac{1}{a\cos\phi}\frac{\partial\psi}{\partial\lambda} + \frac{1}{a}\frac{\partial\chi}{\partial\lambda}\right)$$

For quasi-non-divergent flow ($|\delta| \ll |\zeta|$) it becomes

$$\beta v \approx \frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda}$$

Rossby-Haurwitz Modes

If we suppose that the solution is **quasi-nondivergent** (that is, $|\delta| \ll |\zeta|$), the wind is given approximately in terms of the stream function $(u, v) \approx (-\psi_y, \psi_x)$.

The vorticity equation becomes
$$\nabla^2 \psi_t + \beta \psi_x = O(\delta) \,,$$
and we can ignore the right-hand side.

Assuming the stream function has the wave-like structure of a **spherical harmonic**, we substitute the expression
$$\psi = \psi_0 Y_n^m(\lambda, \phi) \exp(-i\nu t)$$
in the vorticity equation.

We can immediately deduce an expression for the frequency:

$$\nu = \nu_R \equiv -\frac{2\Omega m}{n(n+1)} \,.$$

Repeat
$$\nu = \nu_R \equiv -\frac{2\Omega m}{n(n+1)} \,.$$

This is the celebrated dispersion relation for **Rossby-Haurwitz waves** (Haurwitz, 1940).

$$\star \qquad \star \qquad \star$$

We can ignore sphericity (the β-plane approximation) and assume harmonic dependence
$$\psi(x, y, t) = \psi_0 \exp[i(kx + \ell y - \nu t)] \,,$$

Then the dispersion relation is
$$c = \frac{\nu}{k} = -\frac{\beta}{k^2 + \ell^2} \,,$$
which is the phase-speed found by Rossby (1939).

The Rossby or Rossby-Haurwitz waves are, to the first approximation, **non-divergent waves which travel westward**, the phase speed being greatest for the waves of largest scale.

The RH waves are of relatively low frequency — $|\nu| \leq \Omega$ — and the frequency decreases as the spatial scale decreases.

We may write the divergence equation as
$$\nabla^2 P - f\zeta - \beta\psi_y = O(\delta) \,.$$

Ignoring the r.h.s., we get the **linear balance equation**
$$\nabla^2 P = \nabla \cdot f \nabla \psi \,,$$
a diagnostic relationship between the geopotential and the stream function.

This also follows immediately from the assumption that the wind is both non-divergent and geostrophic:
$$\mathbf{V} = \mathbf{k} \times \nabla\psi \qquad \text{and} \qquad f\mathbf{V} = \mathbf{k} \times \nabla P$$

If variations of f are ignored, we can assume $P = f\psi$. The **wind and pressure are in approximate geostrophic balance** for Rossby-Haurwitz waves.
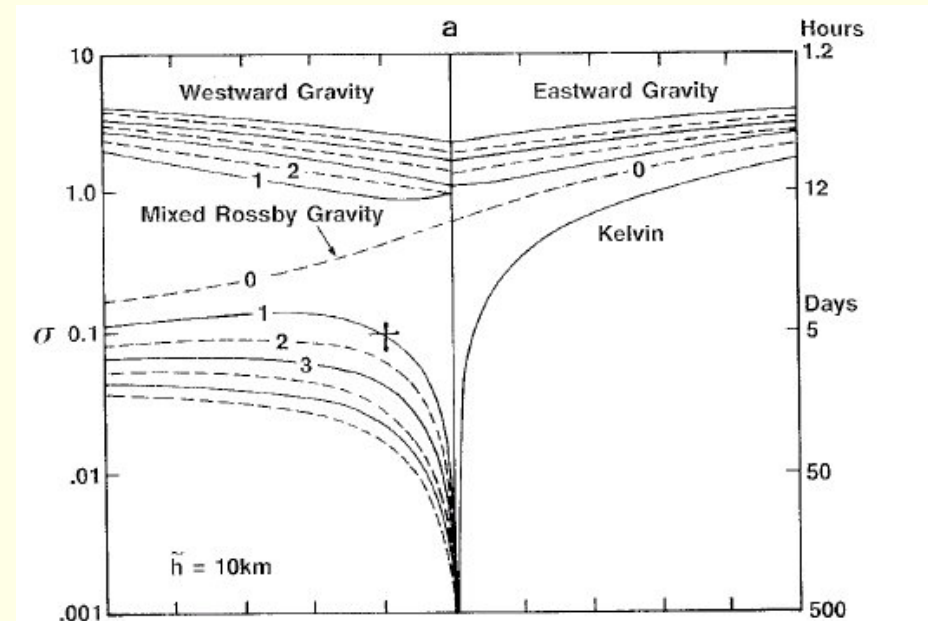
The eigenmodes of the Laplace TIdal Equations ($h = 10\,\text{km}$).

Gravity Wave Modes

We assume now that the solution is **quasi-irrotational**, *i.e.* that $|\zeta| \ll |\delta|$.

Then the wind is given approximately by $(u,v) \approx (\chi_x, \chi_y)$ and the divergence equation becomes

$$\nabla^2 \chi_t + \beta \chi_x + \nabla^2 P = O(\zeta)$$

with the right-hand side negligible.

Using the continuity equation to eliminate P, we get

$$\nabla^2 \chi_{tt} + \beta \chi_{xt} - gh\nabla^4 \chi = 0 \,.$$

We look for a solution of the form

$$\chi = \chi_0 Y_n^m(\lambda, \phi) \exp(-i\nu t)$$

Then we find that

$$\nu^2 + \left(-\frac{2\Omega m}{n(n+1)}\right)\nu - \frac{n(n+1)gh}{a^2} = 0 \,.$$

Repeat

$$\nu^2 + \left(-\frac{2\Omega m}{n(n+1)}\right)\nu - \frac{n(n+1)gh}{a^2} = 0 \,.$$

The coefficient of the second term is just the Rossby-Haurwitz frequency ν_R, so that

$$\nu = \pm\sqrt{\nu_G^2 + (\tfrac{1}{2}\nu_R)^2} - \tfrac{1}{2}\nu_R \,, \qquad \text{where} \qquad \nu_G \equiv \sqrt{\frac{n(n+1)gh}{a^2}} \,,$$
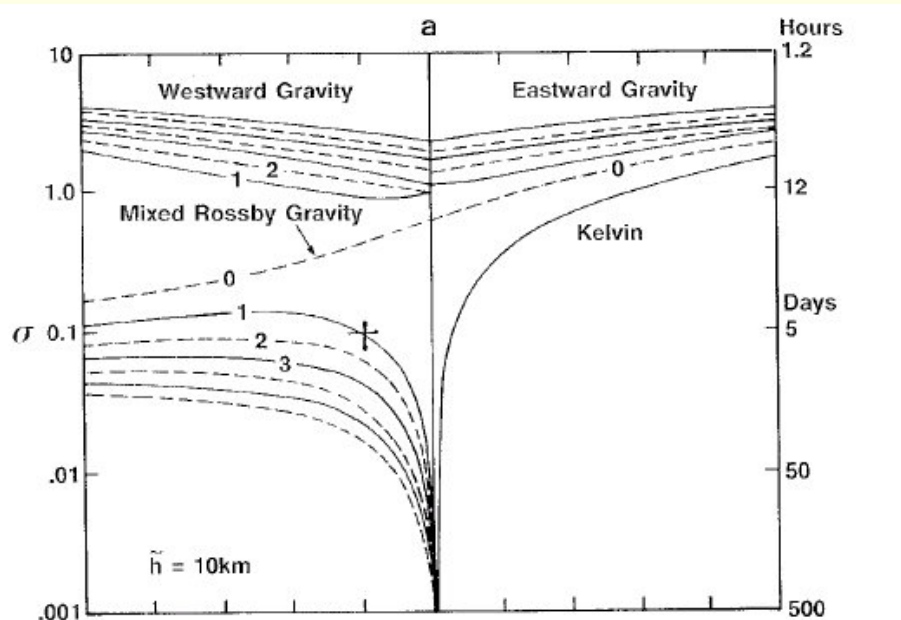
Noting that $|\nu_G| \gg |\nu_R|$, it follows that

$$\nu_\pm \approx \pm\nu_G = \pm\sqrt{\frac{n(n+1)gh}{a^2}} \,,$$

the frequency of pure gravity waves.

There are then two solutions, representing **waves travelling eastward and westward** with equal speeds.

The frequency increases approximately linearly with the total wavenumber n.

The eigenmodes of the Laplace TIdal Equations ($h = 10\,\text{km}$).

Break here

Reminder on linear algebra

Let M be a matrix. An **eigenvector e** of M with eigenvalue λ satisfies

$$\mathbf{Me} = \lambda\mathbf{e}$$

In general there are n eigenvectors for an $n \times n$ matrix.

For a symmetric matrix, the eigenvalues are real and the eigenvectors are orthogonal.

We form the eigenvector and eigenvalue matrices

$$\mathbf{E} = [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n] \quad \text{and} \quad \mathbf{\Lambda} = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$$

Then the eigenvector relationships can be written as

$$\mathbf{ME} = \mathbf{E\Lambda}$$

It follows immediately that

$$\mathbf{M} = \mathbf{E\Lambda E}^{-1} \quad \text{and} \quad \mathbf{E}^{-1}\mathbf{ME} = \mathbf{\Lambda}.$$

Normal Mode Initialization ★

Let X be the state vector of dependent variables.

The model equations can be written schematically as

$$\dot{\mathbf{X}} + i\mathbf{LX} + \mathbf{N}(\mathbf{X}) = \mathbf{0}$$

where L a matrix and N a nonlinear vector function.

Denote the eigenvector matrix of L by E and the diagonal eigenvalue matrix as $\mathbf{\Lambda}$. Then

$$\mathbf{E}^{-1}\mathbf{LE} = \mathbf{\Lambda}.$$

We introduce a transformed state vector $\mathbf{W} = \mathbf{E}^{-1}\mathbf{X}$, and multiply the model equations on the left by $\mathbf{E}^{-1}$.

We get

$$\mathbf{E}^{-1}\dot{\mathbf{X}} + i\mathbf{E}^{-1}\mathbf{LE}\mathbf{E}^{-1}\mathbf{X} + \mathbf{E}^{-1}\mathbf{N}(\mathbf{X}) = \mathbf{0}$$

which may be written

$$\dot{\mathbf{W}} + i\mathbf{\Lambda}\mathbf{W} + \hat{\mathbf{N}}(\mathbf{X}) = \mathbf{0}$$

where $\hat{\mathbf{N}} = \mathbf{E}^{-1}\mathbf{N}(\mathbf{X})$.

The linear system now apearates into two sybsystems. We recall that the eigenvalues fall in to **slow** and **fast** subsets.

We partition the eigenvalue matrix on this basis:

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Lambda_Y} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda_Z} \end{bmatrix}$$

where $\mathbf{\Lambda_Y}$ and $\mathbf{\Lambda_Z}$ are diagonal matrices of eigenfrequencies for the two types of modes.

The state vector W is comprised of two sub-vectors:

$$\mathbf{W} = \begin{bmatrix} \mathbf{Y} \\ \mathbf{Z} \end{bmatrix}$$

The system then separates into two subsystems, for the low and high frequency components:

$$\dot{\mathbf{Y}} + i\mathbf{\Lambda_Y}\mathbf{Y} + \mathbf{N}_Y(\mathbf{Y}, \mathbf{Z}) = \mathbf{0}$$
$$\dot{\mathbf{Z}} + i\mathbf{\Lambda_Z}\mathbf{Z} + \mathbf{N}_Z(\mathbf{Y}, \mathbf{Z}) = \mathbf{0}$$

The vectors Y and Z are the coefficients of the LF and HF components of the flow: the **slow** and **fast** components.

Let us now suppose that the initial fields are separated into slow and fast parts.

The fast modes may be removed so as to leave only the Rossby waves.

$$\text{Replace} \quad \mathbf{W} = \begin{bmatrix} \mathbf{Y} \\ \mathbf{Z} \end{bmatrix} \quad \text{by} \quad \mathbf{W} = \begin{bmatrix} \mathbf{Y} \\ \mathbf{0} \end{bmatrix} \quad \text{at time} \quad t = 0.$$

It might be hoped that this process of **linear normal mode initialization** (imposing the condition $\mathbf{Z} = \mathbf{0}$ at $t = 0$) would ensure a noise-free forecast.

However, the results of the technique are disappointing: the noise is reduced initially, but soon reappears.

The forecasting equations are nonlinear, and the slow components interact nonlinearly in such a way as to generate gravity waves.

The problem of noise remains: the gravity waves are small to begin with, but they grow rapidly.
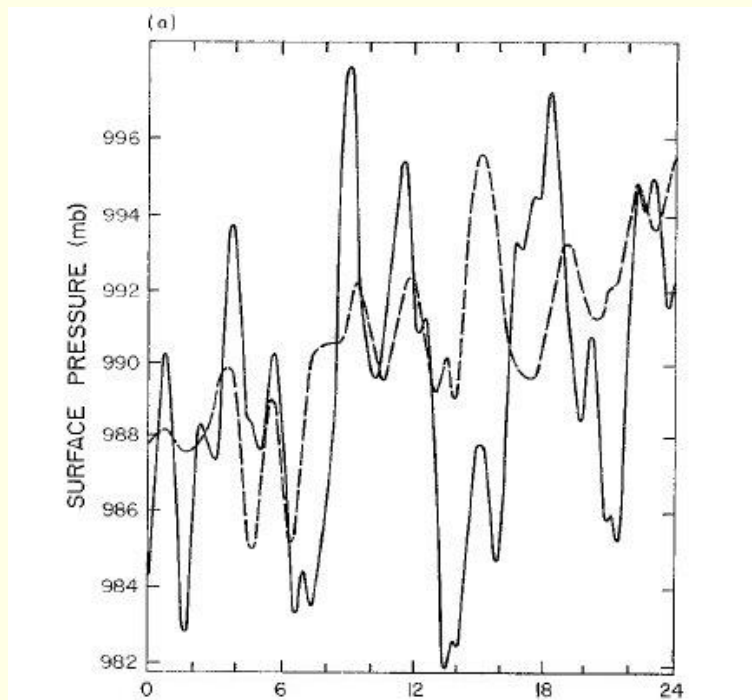
Surface pressure evolution: No Initialization and LNMI.

To control the growth of HF components, Machenhauer (1977) proposed setting their **initial rate-of-change to zero,** in the hope that they would remain small throughout the forecast.

Ferd Baer (1977) proposed a somewhat more general method, using a two-timing perturbation technique.

The forecast, starting from initial fields modified so that $\dot{\mathbf{Z}} = \mathbf{0}$ at $t = 0$ is very smooth and the spurious gravity wave oscillations are almost completely removed.
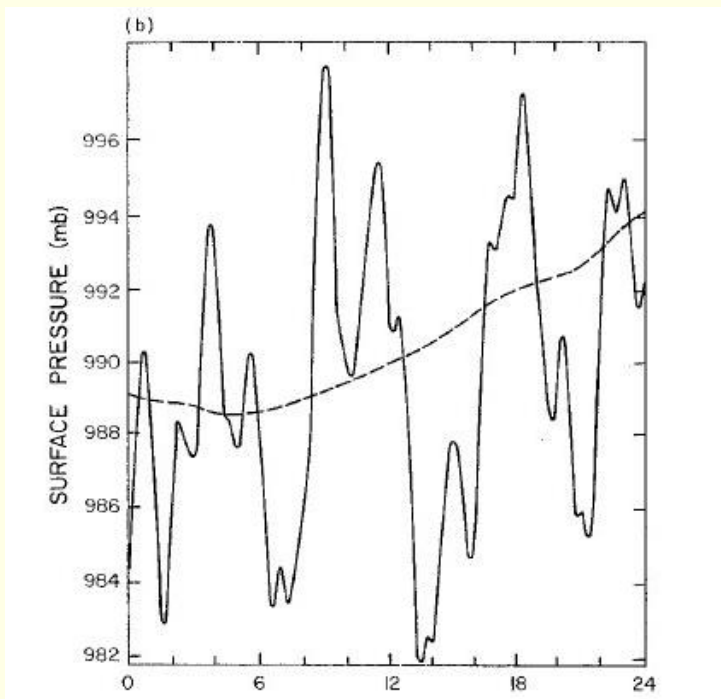
Applying NNMI to the the equation for the fast modes:

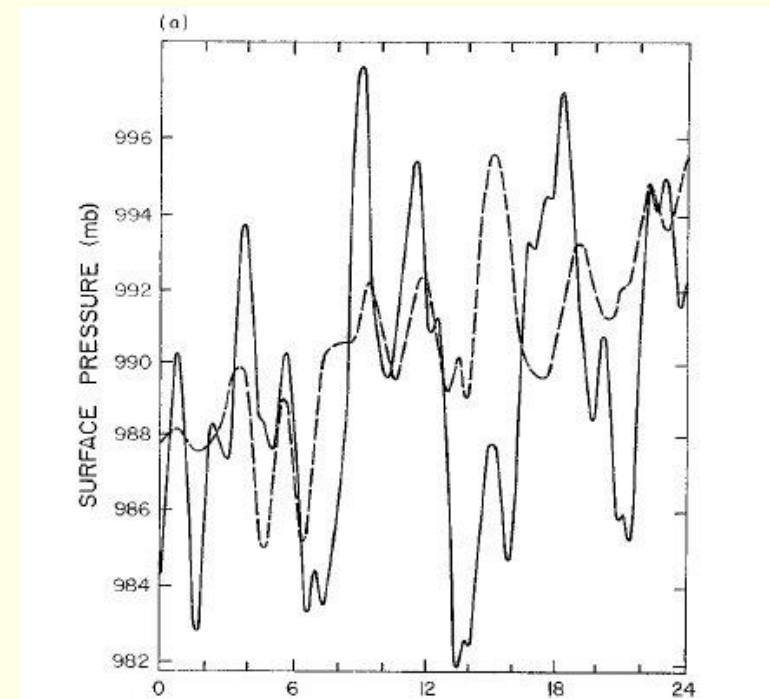$$\dot{\mathbf{Z}} + i\mathbf{\Lambda_Z}\mathbf{Z} + \hat{\mathbf{N}}_Z(\mathbf{Y}, \mathbf{Z}) = \mathbf{0}$$

we get

$$i\mathbf{\Lambda_Z}\mathbf{Z} + \hat{\mathbf{N}}_Z(\mathbf{Y}, \mathbf{Z}) = \mathbf{0} \qquad \text{or} \qquad \mathbf{Z} = i\mathbf{\Lambda_Z^{-1}}\hat{\mathbf{N}}_Z(\mathbf{Y}, \mathbf{Z})$$

The method takes account of the nonlinear nature of the equations, and is referred to as **nonlinear normal mode initialization**.
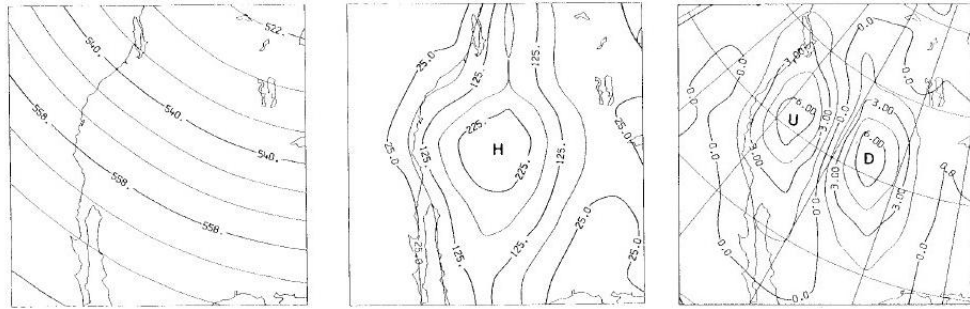
Surface pressure evolution: No Initialization and NNMI.
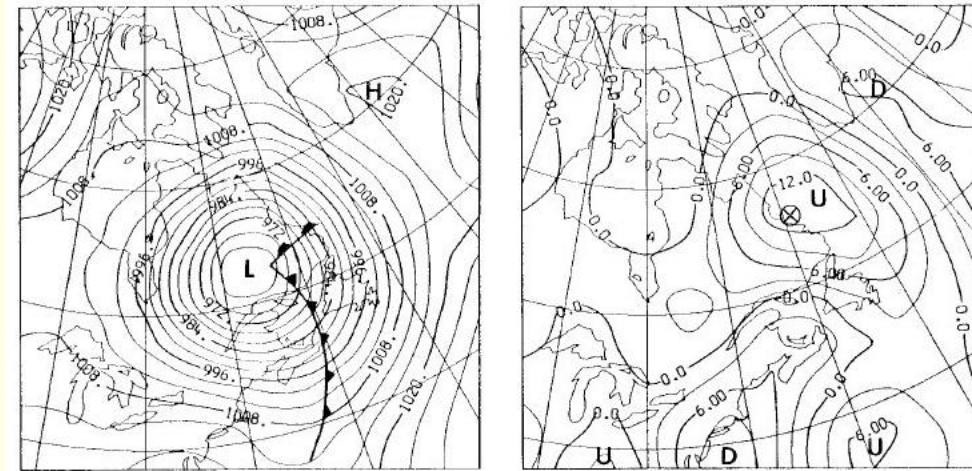

Surface pressure evolution: No Initialization and LNMI.

Vertical velocity for flow over the Rockies.
A realistic w field is generated by nonlinear normal mode initialization.

Generation of vertical velocity in frontal depression.
A realistic w field is generated by nonlinear normal mode initialization.

Example: The Swinging Spring ○

We will illustrate linear normal mode initialization (LNMI) and nonlinear normal mode initialization (NNMI) by application to a simple mechanical system.

The *swinging spring* comprises a heavy bob suspended by a light elastic spring.

The bob is free to move in a vertical plane.

The oscillations of this system are of two types, distinguished by their physical restoring mechanisms.

We consider the elastic oscillations to be analogues of the high frequency gravity waves in the atmosphere.

Similarly, the low frequency rotational motions are considered to correspond to the rotational or Rossby waves.
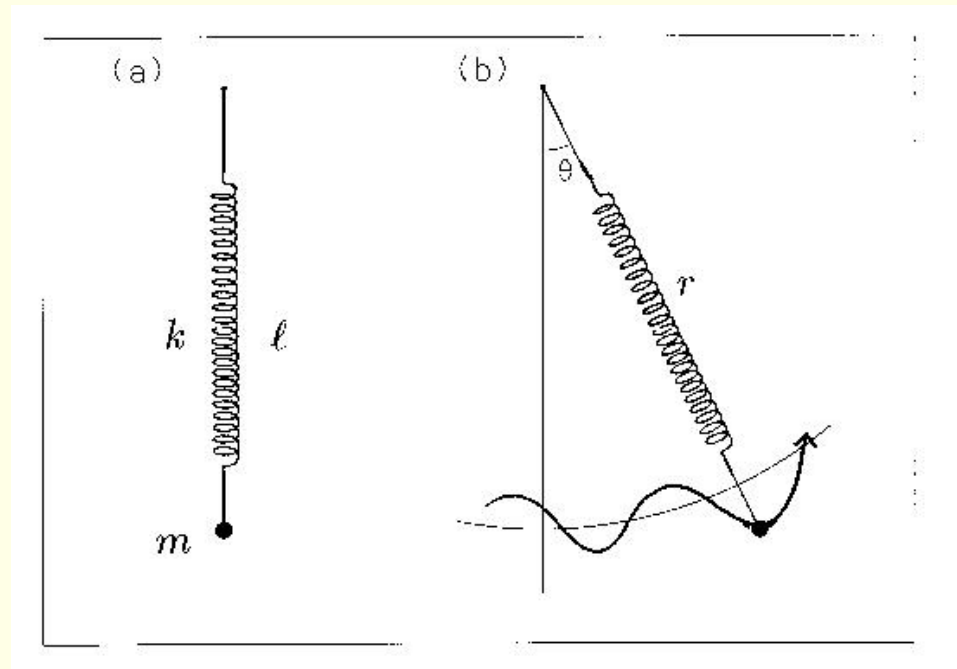
The swinging spring (2D case)

The Dynamical Equations

Let ℓ_0 be the unstretched length of the spring, k its elasticity or stiffness and m the mass of the bob.

At equilibrium the elastic force is balanced by the weight:

$$k(\ell - \ell_0) = mg$$

Polar coordinates $q_r = r$ and $q_\theta = \theta$ are used, and the radial and angular momenta are $p_r = m\dot{r}$ and $p_\theta = mr^2\dot{\theta}$.

The **Hamiltonian** is (in this case) the sum of kinetic, elastic potential and gravitational potential energy:

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \tfrac{1}{2}k(r-\ell_0)^2 - mgr\cos\theta\,.$$

(If the Hamiltonian formalism is unfamiliar, the equations may be derived by considering the forces on the bob).

The (canonical) dynamical equations may now be written

$$\begin{aligned}
\dot{\theta} &= p_\theta/mr^2 \\
\dot{p}_\theta &= -mgr\sin\theta \\
\dot{r} &= p_r/m \\
\dot{p}_r &= p_\theta^2/mr^3 - k(r-\ell_0) + mg\cos\theta\,.
\end{aligned}$$

These equations may be written symbolically in vector form

$$\dot{\mathbf{X}} + \mathbf{L}\mathbf{X} + \mathbf{N}(\mathbf{X}) = 0$$

where $\mathbf{X} = (\theta, p_\theta, r, p_r)^{\mathrm{T}}$, $\mathbf{L}$ is the matrix of coefficients of the linear terms and $\mathbf{N}$ is a nonlinear vector function.

We now suppose that the amplitude of the motion is small, so that $|r'| = |r - \ell| \ll \ell$ and $|\theta| \ll 1$.

The state vector $\mathbf{X}$ comprises two sub-vectors:

$$\mathbf{X} = \begin{pmatrix}\mathbf{Y}\\\mathbf{Z}\end{pmatrix}, \qquad \text{where} \quad \mathbf{Y} = \begin{pmatrix}\theta\\p_\theta\end{pmatrix} \quad \text{and} \quad \mathbf{Z} = \begin{pmatrix}r'\\p_r\end{pmatrix},$$

The linear slow and fast modes evolve independently.

We call the motion described by $\mathbf{Y}$ the **rotational component** and that described by $\mathbf{Z}$ the **elastic component**.

The rotational equations may be written

$$\ddot{\theta} + (g/\ell)\theta = 0$$

which is the equation for a **simple pendulum** having oscillatory solutions with frequency

$$\omega_{\mathrm{R}} \equiv \sqrt{\frac{g}{\ell}}\,.$$

The remaining two equations yield

$$\ddot{r}' + (k/m)r' = 0\,,$$

the equations for **elastic oscillations** with frequency

$$\omega_{\mathrm{E}} = \sqrt{\frac{k}{m}}\,.$$

We define the **ratio of the rotational and elastic frequencies**:

$$\omega_{\mathrm{R}} = \sqrt{\frac{g}{\ell}}, \qquad \omega_{\mathrm{E}} = \sqrt{\frac{k}{m}}, \qquad \epsilon \equiv \left(\frac{\omega_{\mathrm{R}}}{\omega_{\mathrm{E}}}\right).$$

It is easily shown that $\epsilon < 1$, so the rotational frequency is always less than the elastic.

$$\epsilon^2 = \frac{mg}{k\ell} = \left(1 - \frac{\ell_0}{\ell}\right) < 1\,, \qquad \text{so that} \qquad |\omega_{\mathrm{R}}| < |\omega_{\mathrm{E}}|\,.$$

We assume that the parameters are such that

$$\epsilon \ll 1$$

In this case the linear normal modes are clearly distinct:

- The rotational mode has low frequency (**LF**)
- The elastic mode has high frequency (**HF**).

Linear and Nonlinear Initialization

For small amplitude motions the LF and HF oscillations are completely independent of each other.

They evolve without interaction.

We can suppress the HF component completely by setting its initial amplitude to zero:

$$\mathbf{Z} = (r', p_r)^{\mathrm{T}} = \mathbf{0} \quad \text{at} \quad t = 0\,.$$

This procedure is called **linear initialization**.

When the amplitude is large, nonlinear terms are no longer negligible. **The LF and HF motions interact**.

It is clear from the equations that linear initialization will not ensure permanent absence of HF motions ...

... the nonlinear LF terms generate radial momentum.

To achieve better results, we set the **initial tendency** of the HF components to zero:

$$\dot{\mathbf{Z}} = (\dot{r}, \dot{p}_r)^{\mathrm{T}} = \mathbf{0} \quad \text{at} \quad t = 0\,,$$

This procedure is called **nonlinear initialization**.

For the spring, we can deduce **explicit expressions** for the initial conditions:

$$r(0) = r_{\mathrm{B}} \equiv \frac{\ell(1 - \epsilon^2(1 - \cos\theta))}{1 - (\dot{\theta}/\omega_{\mathrm{E}})^2}\,, \qquad p_r(0) = 0\,.$$
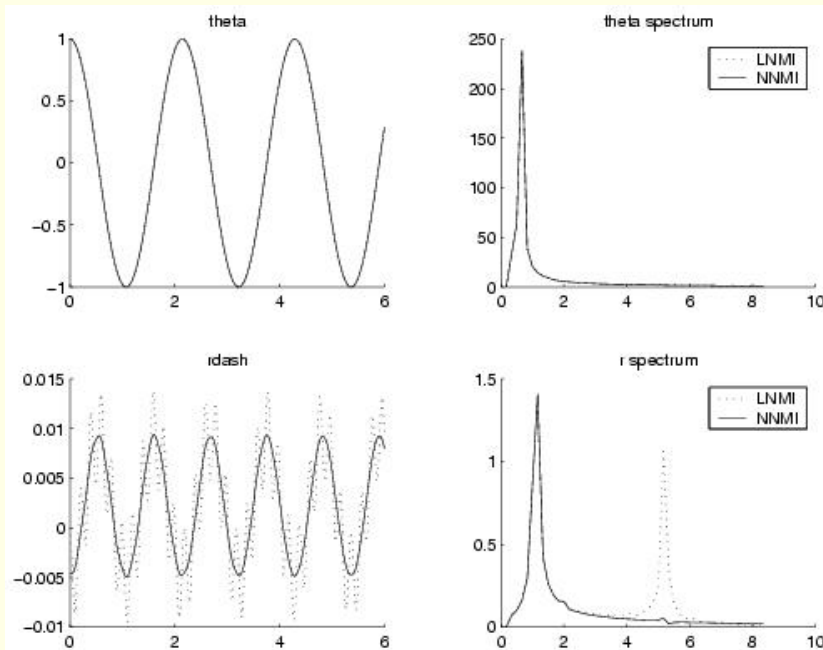
Thus, given arbitrary initial conditions $\mathbf{X} = (\theta, p_\theta, r, p_r)^{\mathrm{T}}$,

$$\text{Replace} \qquad \mathbf{Z} = (r, p_r)^{\mathrm{T}} \qquad \text{by} \qquad \mathbf{Z}_{\mathrm{B}} = (r_{\mathrm{B}}, 0)^{\mathrm{T}}\,.$$

The rotational component $\mathbf{Y} = (\theta, p_\theta)^{\mathrm{T}}$ remains unchanged.

Does it work? An example to follow shows that it does!

Solution of swinging spring equations for linear (LNMI) and nonlinear (NNMI) initialization.

A Numerical Example

In the accompanying figure, we show the results of **two integrations** of the spring equations.
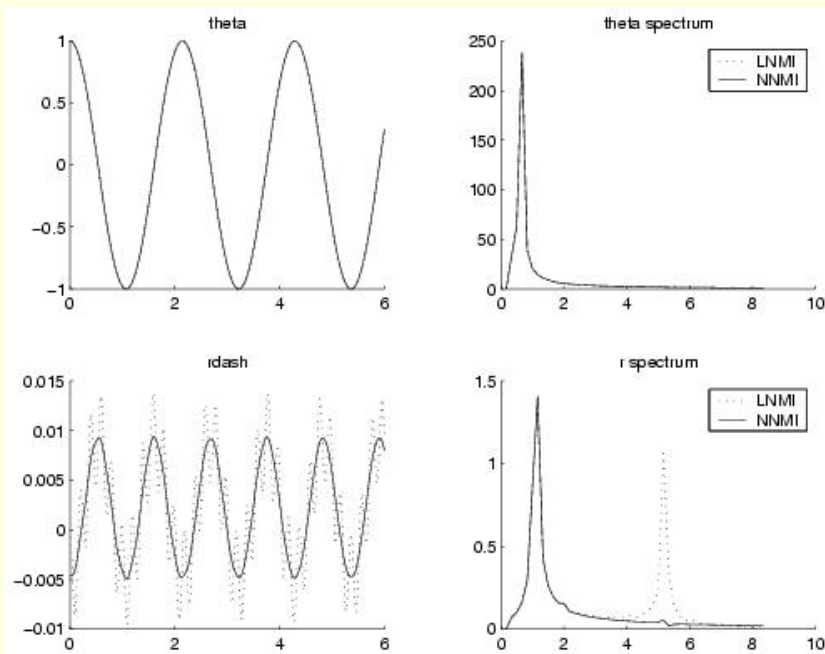
The parameter values are

$$m = 1\,, \quad g = \pi^2\,, \quad k = 100\pi^2 \quad \text{and} \quad \ell = 1 \quad \text{(SI units)}$$

Thus, $\epsilon = 0.1$ and the periods of the swinging and springing motions are respectively

$$\tau_{\mathrm{R}} = 2\,\mathrm{s} \qquad \text{and} \qquad \tau_{\mathrm{E}} = 0.2\,\mathrm{s}\,.$$

The initial conditions are vanishing velocity ($\dot{r} = \dot{\theta} = 0$), with $\theta(0) = 1$ and $r(0) \in \{1, 0.99540\}$.

The equations are integrated over a period of 6 seconds

Solution of swinging spring equations for linear (LNMI) and nonlinear (NNMI) initialization.

The upper panels show the evolution and spectrum of the slow variable θ

The lower panels are for the fast variable r.

Dotted curves are for linear initialization and solid curves for nonlinear initialization.

For the slow variable, the curves are indistinguishable.

The spectrum has a clear peak at a frequency of 0.5 cycles per second (Hz).

For the fast variable, the linearly initialized evolution has high frequency noise (dotted curve, lower left panel).

This is confirmed in the spectrum: there is a sharp peak at 5 Hz.

When nonlinearly initialized, this peak is removed: only the peak at 1 Hz remains.

This is the 'balanced fast motion'.

The 'balanced fast motion' can be understood physically . . .

. . . The centrifugal effect stretches the spring twice for each pendular swing: the result is a component of r with a period of one second.

The radial variation does not disappear for balanced motion, but it is of low frequency.

The balanced fast motion is said to be 'slaved' (or, better, enslaved) to the slow motion.
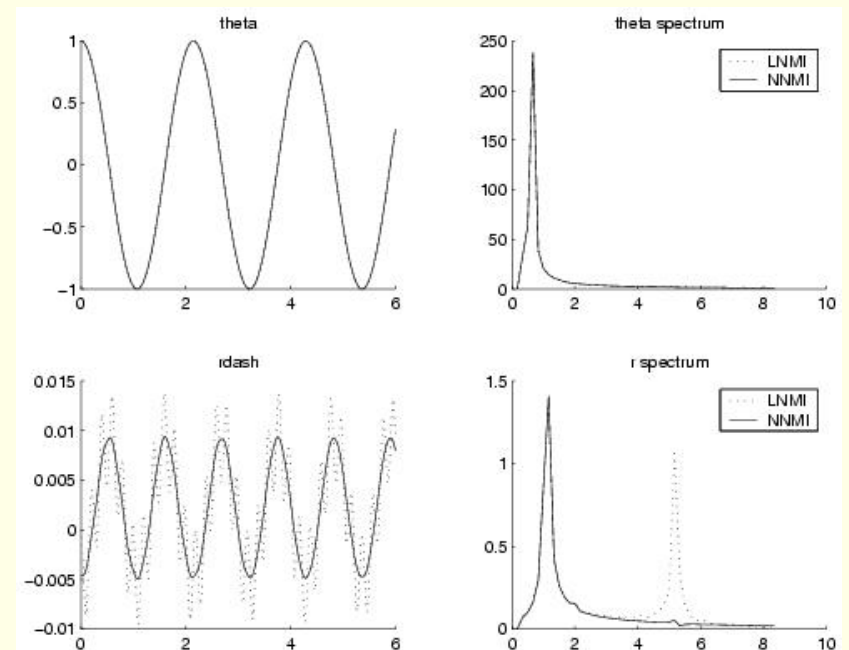
Solution of swinging spring equations for linear (LNMI) and nonlinear (NNMI) initialization.

End of §4.2