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- **In numerical weather prediction (NWP), timeliness of the forecast is of the essence.**

- **In this lecture, we study an alternative approach to time integration, which is unconditionally stable and so, free from the shackles of the CFL condition.**

The Basic Idea

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The problem with this aproach, is that the distribution of representative parcels rapidly becomes *highly non-uniform*.

In the *semi-Lagrangian scheme* the individual parcels are followed only for a single time-step. After each step, we revert to a uniform grid.

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The first *operational implementation* of a semi-Lagrangian scheme was in 1982 at the Irish Meteorological Service.

Semi-Lagrangian advection schemes are now in widespread use in all the main Numerical Weather Prediction centres.

Multiply-Upstream, Semi-Lagrangian Advective Schemes: Analysis and Application to a Multi-Level Primitive Equation Model

J. R. BATES AND A. MCDONALD

Irish Meteorological Service, Dublin, Ireland

(Manuscript received 12 April 1982, in final form 16 September 1982)

ABSTRACT

The stability properties of some simple semi-Lagrangian advective schemes, based on a multiply-upstream interpolation, are examined. In these schemes, the interpolation points are chosen to surround the departure points of the fluid particles at the beginning of a time step. It is shown that the schemes, though explicit, are unconditionally stable for a constant wind field.

Application of the schemes to a multi-level split explicit model shows that they enable full advantage to be taken of the splitting method by allowing a long time step for advection. It is shown that they can thus lead to a considerable saving of computer time compared to Eulerian schemes, while giving comparable accuracy.

Paper in *Monthly Weather Review*, 1982.

Eulerian and Lagrangian Approach

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To develop numerical solution methods, we may start from ***either*** **the Eulerian** ***or*** **the Lagrangian form of the equation.**

For the semi-Lagrangian scheme, we choose the latter.

Since the advection equation is linear, we can construct a general solution from Fourier components

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Therefore, in analysing the properties of numerical schemes, we seek a solution of the form

$$Y_m^n = a \times \exp(-i\omega n\Delta t) \times \exp(ikm\Delta x) = aA^n\exp(ikm\Delta x)$$

where $A = \exp(-i\omega\Delta t)$**.**

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The character of the solution depends on the modulus of A**:**

If $|A| < 1$**, the solution *decays* with time.**

If $|A| = 1$**, the solution is *neutral* with time.**

If $|A| > 1$**, the solution *grows* with time.**

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In the third case (growing solution), the scheme is *unstable*.

Numerical Domain of Dependence.

Space axis horizontal
Time axis vertical

```
+--------+--------+--------●--------+--------+--------+ n
|        |        |     *******     |        |        |
|        |        |  *************  |        |        |
+--------+--------*******************--------+--------+ n-1
|        |     *************************     |        |
|        |  *******************************  |        |
+--------*************************************--------+ n-2
|     *******************************************     |
|  *************************************************  |
******************************************************* n-3
|        |        |        |        |        |        |

m-3     m-2      m-1       m       m+1      m+2      m+3
```

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```
+---------+---------+---------●---------+---------+---------+ n
|         |         |      *******      |         |         |
|         |         |   *************   |         |         |
+---------+---------*********************---------+---------+ n-1
|         |      ***************************      |         |
|         |   *********************************   |         |
+---------*****************************************---------+ n-2
|      ***********************************************      |
|   *****************************************************   |
************************************************************* n-3
|         |         |         |         |         |         |

m-3      m-2       m-1        m        m+1       m+2       m+3
```

For the **Eulerian Leapfrom Scheme**, the value Y_m^n at time $n\Delta t$ and position $m\Delta x$ depends on values within the area depicted by asterisks.

Values outside this region have *no influence* on Y_m^n.

Numerical Domain of Dependence

Each computed value Y_m^n depends on previously computed values and on the initial conditions. The set of points which influence the value Y_m^n is called the *numerical domain of dependence* of Y_m^n.

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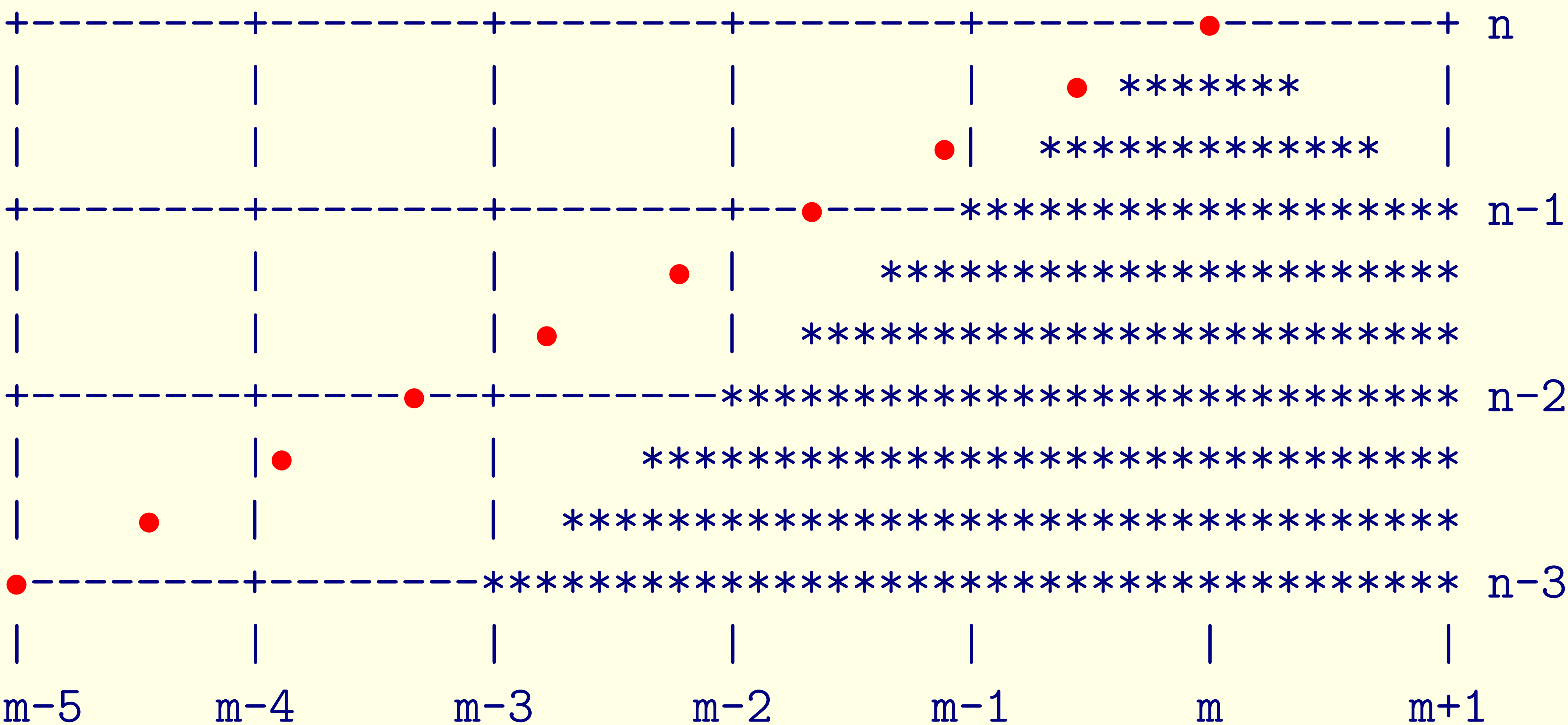
Worse again, the numerical solution may bear absolutely no relationship to the physical solution and may grow exponentially with time even when the true solution is bounded.

A *necessary condition* for avoidance of this phenomenon is that the numerical domain of dependence should include the physical trajectory. This condition is fulfilled by the semi-Lagrangian scheme.

Parcel coming from Outside Domain of Dependence



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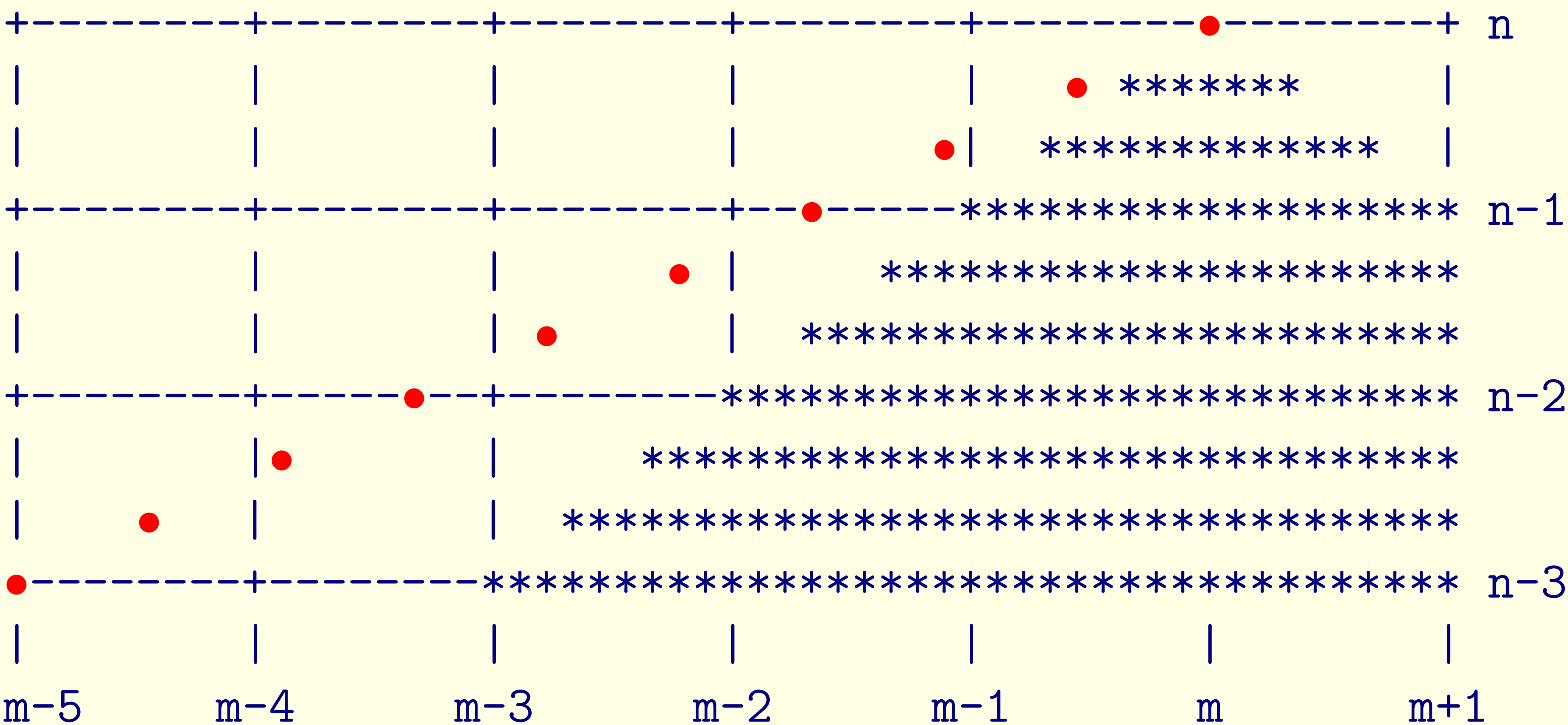

The line of bullets (•) represents a parcel trajectory ($\mu = \frac{5}{3}$).

The value everywhere on the trajectory is Y_m^n. ($c = 5\Delta x / 3\Delta t$).

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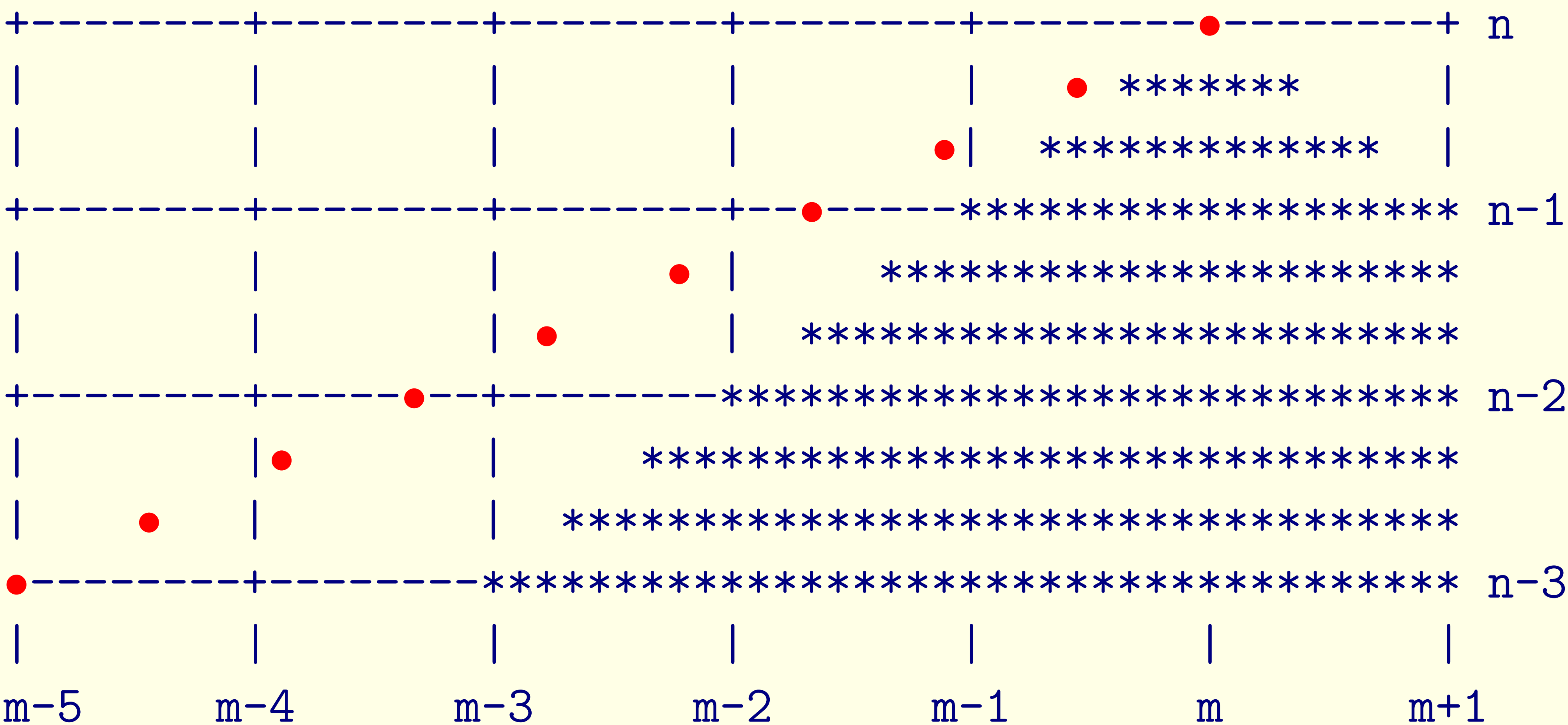

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The value everywhere on the trajectory is Y_m^n. ($c = 5\Delta x / 3\Delta t$).

Since the parcel originates *outside* the numerical domain of dependence, the Eulerian scheme *cannot* model it correctly.

The central idea of the Lagrangian scheme is to ***represent the physical trajectory of the fluid parcel*****.**

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We consider a parcel *arriving* at gridpoint $m\Delta x$ at the new time $(n+1)\Delta t$ and ask: Where has it come from?

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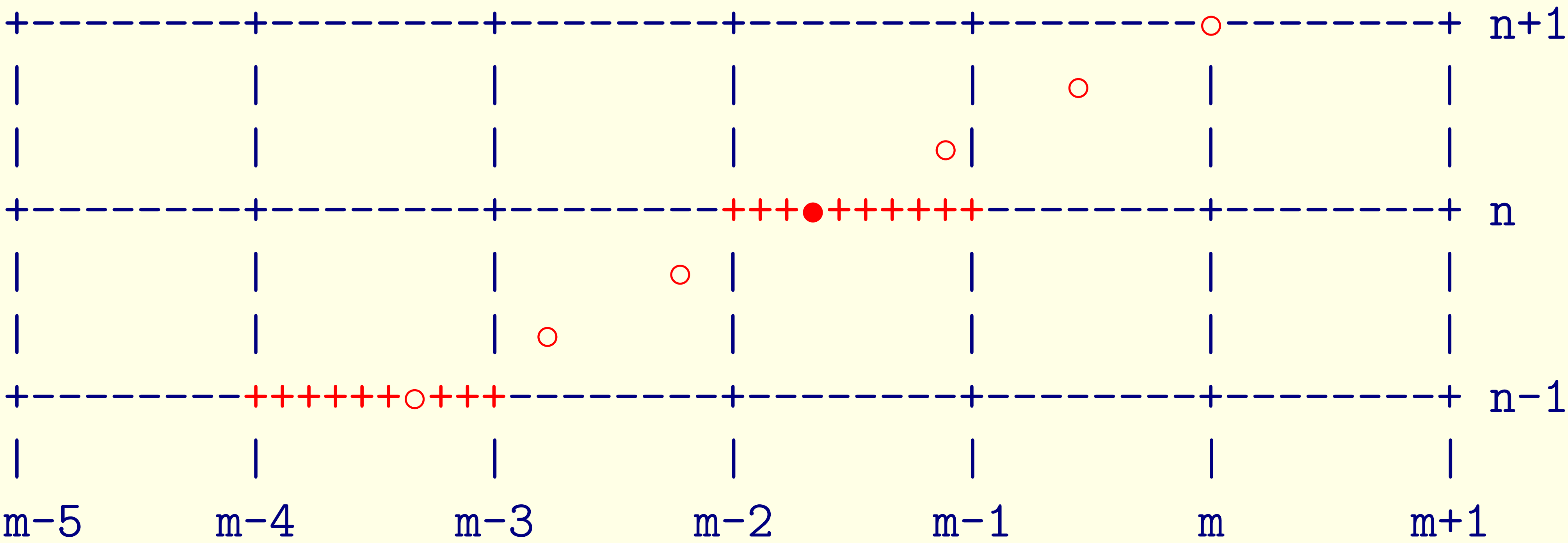
But this interpolation ensures that the trajectory falls within the numerical domain of dependence.

We will show that this leads to a *numerically stable scheme*.

Interpolation using Surrounding Points



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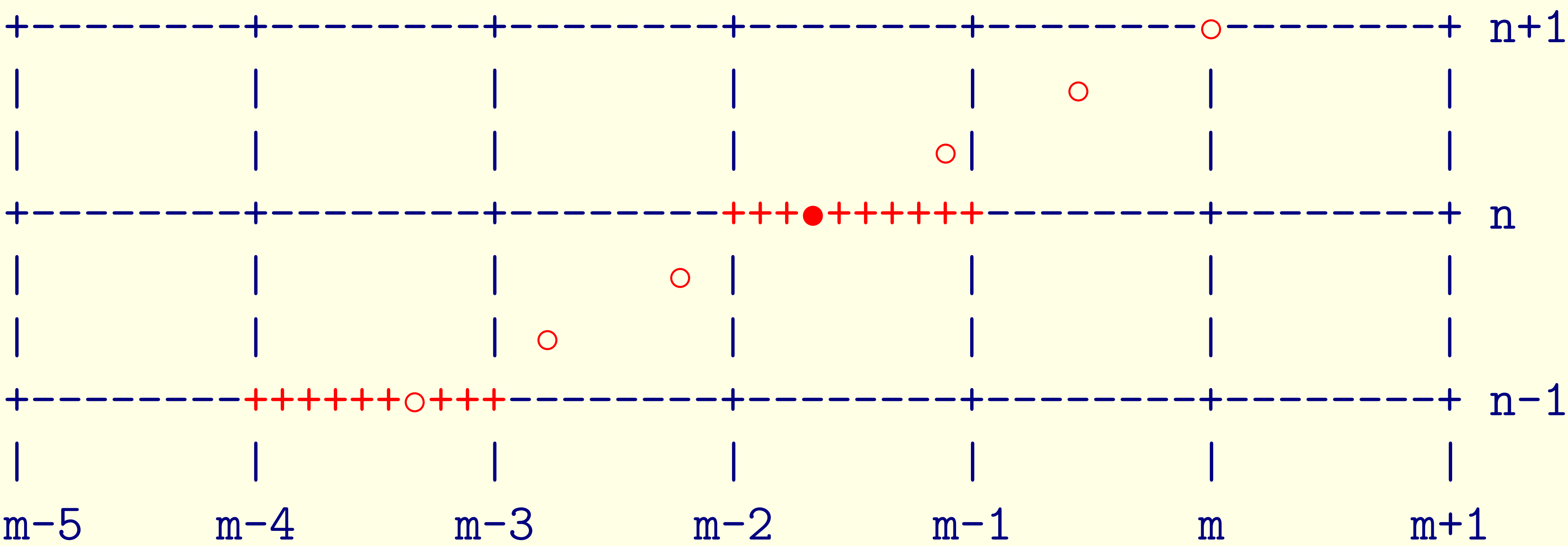

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At time $n\Delta t$ the parcel is at (●), which is not a grid-point.

Interpolation using Surrounding Points

```
+---------+---------+---------+---------+---------○---------+ n+1
|         |         |         |         |   ○     |         |
|         |         |         |        ○|         |         |
+---------+---------+---------+++●+++++++---------+---------+ n
|         |         |       ○ |         |         |         |
|         |         | ○       |         |         |         |
+---------+++++++○+++---------+---------+---------+---------+ n-1
|         |         |         |         |         |         |
m-5       m-4       m-3       m-2       m-1       m         m+1
```

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At time $n\Delta t$ the parcel is at (●), which is not a grid-point.

The value at the departure point is obtained by interpolation from surrounding points.

Thus we ensure that, even though $\mu = \frac{5}{3} > 1$, the physical trajectory is *within* the domain of numerical dependence.

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Applying the equation over the time interval $[n\Delta t, (n+1)\Delta t]$, we get

$$\begin{pmatrix}\textbf{Value of } Y \textbf{ at} \\ \textbf{point } m\Delta x \textbf{ at} \\ \textbf{time } (n+1)\Delta t\end{pmatrix} = \begin{pmatrix}\textbf{Value of } Y \textbf{ at} \\ \textbf{\textit{departure point}} \\ \textbf{at time } n\Delta t\end{pmatrix}$$

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In a more compact form, we may write

$$Y_m^{n+1} = Y_\bullet^n$$

where $Y_\bullet^n$ represents the value at the departure point, which is normally not a grid point.

Interpolation using Surrounding Points

```
+---------+---------+---------+---------+---------o---------+ n+1
|         |         |         |         |    o    |         |
|         |         |         |        o|         |         |
+---------+---------+---------+++●++++++---------+---------+ n
m-5       m-4       m-3       m-2       m-1       m         m+1
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The distance travelled in time Δt **is** $s = c\Delta t$**.**

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```
+---------+---------+---------+---------+---------o---------+ n+1
|         |         |         |         |    o    |         |
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m-5      m-4       m-3       m-2       m-1        m        m+1
```

The distance travelled in time Δt is $s = c\Delta t$.
The Courant Number is $\mu = \frac{c\Delta t}{\Delta x}$. Here, $\mu = \frac{5}{3}$. We define:

$$p = [\mu] = \text{ Integral part of } \mu$$
$$\alpha = \mu - p = \text{ Fractional part of } \mu$$

Note that, by definition, $0 \leq \alpha < 1$ (here, $p = 1$ and $\alpha = 2/3$).

Interpolation using Surrounding Points

```
+---------+---------+---------+---------+---------o---------+ n+1
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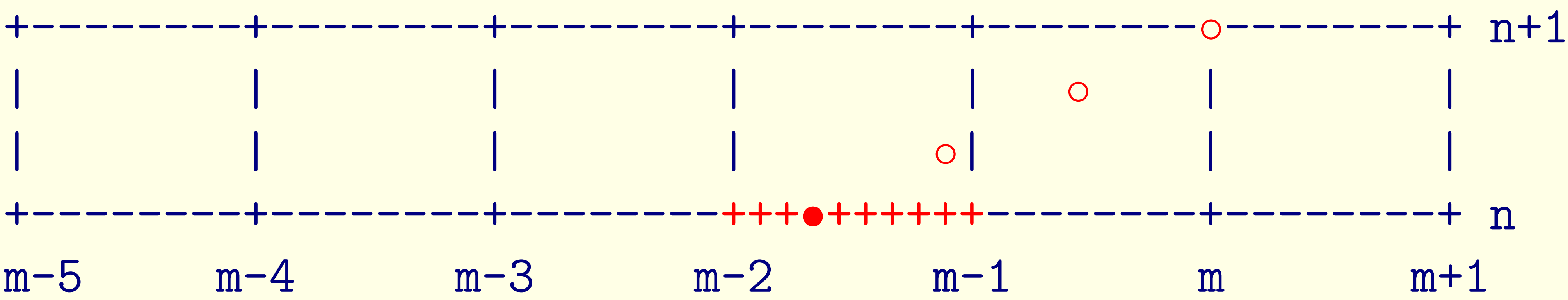

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So, the departure point falls between the grid points $m - p - 1$ and $m - p$.
A *linear interpolation* gives

$$Y_{\bullet}^{n} = \alpha Y_{m-p-1}^{n} + (1 - \alpha) Y_{m-p}^{n} \,.$$

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```
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|         |         |         |         |    o    |         |
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+---------+---------+---------+++●+++++++---------+---------+ n

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$$
Y^n_{\bullet} = \alpha Y^n_{m-p-1} + (1 - \alpha) Y^n_{m-p} \, .
$$

Check: Show what this implies in the limits $\alpha = 0$ and $\alpha \to 1$.

Break here

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We can write this as

$$A = \exp(-ikp\Delta x) \cdot [(1-\alpha) + \alpha \exp(-ik\Delta x)]$$

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Now consider the squared modulus of A:

$$\begin{aligned}
|A|^2 &= |\exp(-ikp\Delta x)|^2 \cdot |(1-\alpha) + \alpha \exp(-ik\Delta x)|^2 \\
&= |(1-\alpha) + \alpha \cos k\Delta x - i\alpha \sin k\Delta x|^2 \\
&= [(1-\alpha) + \alpha \cos k\Delta x]^2 + \alpha[\sin k\Delta x]^2 \\
&= (1-\alpha)^2 + 2(1-\alpha)\alpha \cos k\Delta x + \alpha^2 \cos^2 k\Delta x + \alpha^2 \sin^2 k\Delta x \\
&= (1 - 2\alpha + \alpha^2) + 2\alpha(1-\alpha) \cos k\Delta x + \alpha^2 \\
&= 1 - 2\alpha(1-\alpha)[1 - \cos k\Delta x]\,.
\end{aligned}$$

Again,

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In either case, $|A|^2 \le 1$, so ***there is numerical stability*****.**

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- ***This is a substantial gain in computational efficiency.***

$\star \quad \star \quad \star$

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End of §3.2.6