§3.2.6. Semi-Lagrangian Advection

- We have studied the Eulerian *leapfrog scheme* and found it to be conditionally stable.
- The criterion for stability was the CFL condition

$$\mu \equiv \frac{c\Delta t}{\Delta x} \leq 1 \, .$$

- For high spatial resolution (small Δx) this severly limits the maximum time step Δt that is allowed.
- In numerical weather prediction (NWP), timeliness of the forecast is of the essence.
- In this lecture, we study an alternative approach to time integration, which is unconditionally stable and so, free from the shackles of the CFL condition.

The Basic Idea

The semi-Lagrangian scheme for advection is based on the idea of approximating the Lagrangian time derivative.

It is so formulated that ***the numerical domain of dependence always includes the physical domain of dependence***. This necessary condition for stability is satisfied automatically by the scheme.

In a ***fully Lagrangian*** scheme, the trajectories of actual physical parcels of fluid would be followed throughout the motion.

The problem with this aproach, is that the distribution of representative parcels rapidly becomes ***highly non-uniform***.

In the ***semi-Lagrangian scheme*** the individual parcels are followed only for a single time-step. After each step, we revert to a uniform grid.

The semi-Lagrangian algorithm has enabled us to integrate the primitive equations using a time step of 15 minutes.

This can be compared to a typical timestep of 2.5 minutes for conventional schemes.

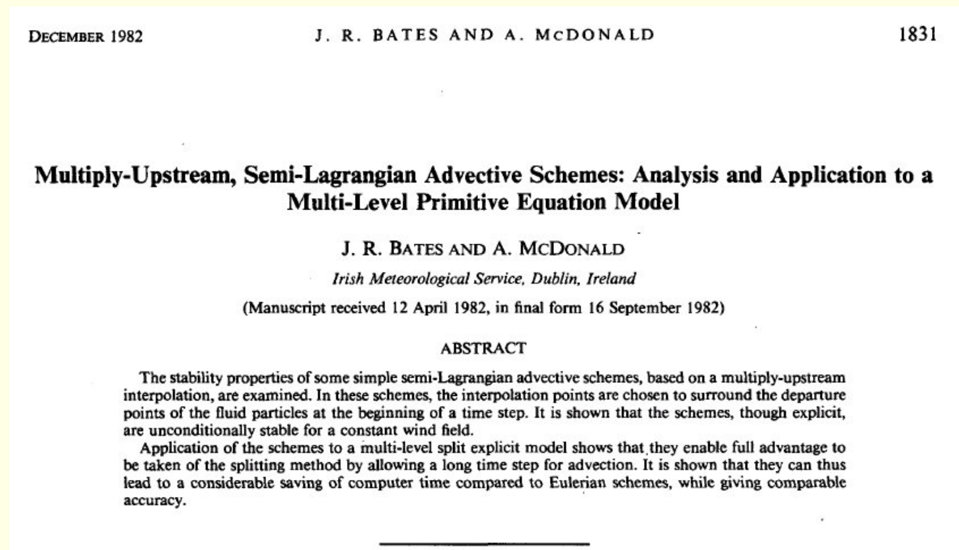
The consequential saving of computation time means that the operational numerical guidance is available to the forecasters much earlier than would otherwise be the case.

The semi-Lagrangian method was pioneered by the renowned Canadian meteorologist André Robert.

Robert also popularized the semi-implicit method.

The first *operational implementation* of a semi-Lagrangian scheme was in 1982 at the Irish Meteorological Service.

Semi-Lagrangian advection schemes are now in widespread use in all the main Numerical Weather Prediction centres.



DECEMBER 1982 J. R. BATES AND A. McDONALD 1831

Multiply-Upstream, Semi-Lagrangian Advective Schemes: Analysis and Application to a Multi-Level Primitive Equation Model

J. R. BATES AND A. McDONALD

Irish Meteorological Service, Dublin, Ireland

(Manuscript received 12 April 1982, in final form 16 September 1982)

ABSTRACT

The stability properties of some simple semi-Lagrangian advective schemes, based on a multiply-upstream interpolation, are examined. In these schemes, the interpolation points are chosen to surround the departure points of the fluid particles at the beginning of a time step. It is shown that the schemes, though explicit, are unconditionally stable for a constant wind field.

Application of the schemes to a multi-level split explicit model shows that they enable full advantage to be taken of the splitting method by allowing a long time step for advection. It is shown that they can thus lead to a considerable saving of computer time compared to Eulerian schemes, while giving comparable accuracy.

Paper in *Monthly Weather Review*, 1982.

Eulerian and Lagrangian Approach

We consider the *linear advection equation* which describes the conservation of a quantity $Y(x,t)$ following the motion of a fluid flow in one dimension with constant velocity c.

This may be written in either of two alternative forms:

$$\frac{\partial Y}{\partial t} + c\frac{\partial Y}{\partial x} = 0 \qquad \Leftarrow \qquad \text{Eulerian Form}$$

$$\frac{dY}{dt} = 0 \qquad \Leftarrow \qquad \text{Lagrangian Form}$$

The general solution is $Y = Y(x - ct)$.

To develop numerical solution methods, we may start from *either* the Eulerian *or* the Lagrangian form of the equation.

For the semi-Lagrangian scheme, we choose the latter.

Since the advection equation is linear, we can construct a general solution from Fourier components

$$Y = a\exp[ik(x - ct)]\,; \qquad k = 2\pi/L\,.$$

This expression may be separated into the product of a function of space and a function of time:

$$Y = a \times \exp(-i\omega t) \times \exp(ikx)\,; \qquad \omega = kc\,.$$

Therefore, in analysing the properties of numerical schemes, we seek a solution of the form

$$Y_m^n = a \times \exp(-i\omega n\Delta t) \times \exp(ikm\Delta x) = aA^n\exp(ikm\Delta x)$$

where $A = \exp(-i\omega\Delta t)$.
The character of the solution depends on the modulus of A:

If $|A| < 1$, the solution ***decays*** with time.
If $|A| = 1$, the solution is ***neutral*** with time.
If $|A| > 1$, the solution ***grows*** with time.

In the third case (growing solution), the scheme is ***unstable***.

Numerical Domain of Dependence.

Space axis horizontal
Time axis vertical

```
+--------+--------+--------●--------+--------+--------+ n
|        |        |     *******     |        |        |
|        |        |  *************  |        |        |
+--------+--------*******************--------+--------+ n-1
|        |     *************************     |        |
|        |  *******************************  |        |
+--------*************************************--------+ n-2
|     *******************************************     |
|  *************************************************  |
******************************************************* n-3
|        |        |        |        |        |        |
m-3      m-2      m-1      m        m+1      m+2      m+3
```

For the Eulerian Leapfrom Scheme, the value Y_m^n at time $n\Delta t$ and position $m\Delta x$ depends on values within the area depicted by asterisks.

Values outside this region have ***no influence*** on Y_m^n.

Numerical Domain of Dependence

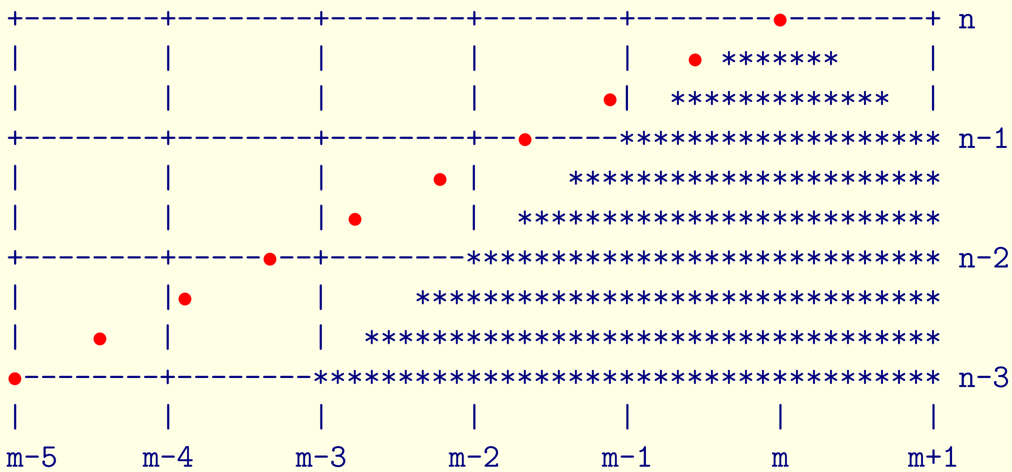
Each computed value Y_m^n depends on previously computed values and on the initial conditions. The set of points which influence the value Y_m^n is called the ***numerical domain of dependence*** of Y_m^n.

It is clear on physical grounds that if the parcel of fluid arriving at point $m\Delta x$ at time $n\Delta t$ originates ***outside the numerical domain of dependence***, the numerical scheme cannot yield an accurate result: the necessary information is not available to the scheme.

Worse again, the numerical solution may bear absolutely no relationship to the physical solution and may grow exponentially with time even when the true solution is bounded.

A ***necessary condition*** for avoidance of this phenomenon is that the numerical domain of dependence should include the physical trajectory. This condition is fulfilled by the semi-Lagrangian scheme.

Parcel coming from Outside Domain of Dependence

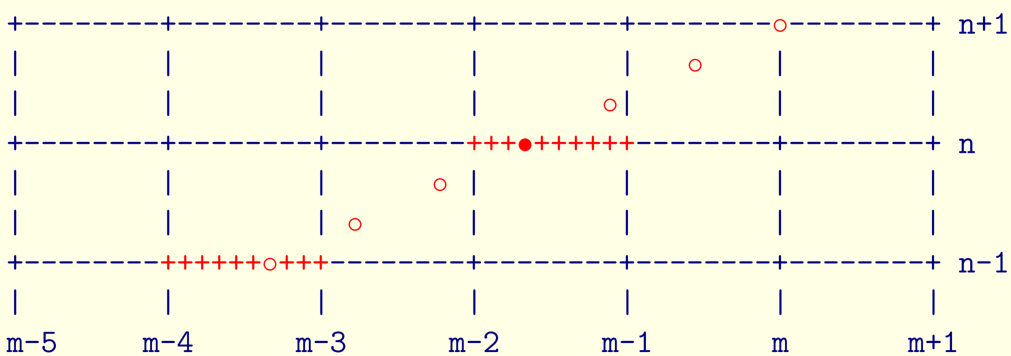


The line of bullets (•) represents a parcel trajectory ($\mu = \frac{5}{3}$).

The value everywhere on the trajectory is Y^n_m. ($c = 5\Delta x / 3\Delta t$).

Since the parcel originates *outside* the numerical domain of dependence, the Eulerian scheme *cannot* model it correctly.

Interpolation using Surrounding Points



The line of circles (○) represents a parcel trajectory ($c = \frac{5\Delta x}{3\Delta t}$)

At time $n\Delta t$ the parcel is at (•), which is not a grid-point.

The value at the departure point is obtained by interpolation from surrounding points.

Thus we ensure that, even though $\mu = \frac{5}{3} > 1$, the physical trajectory is *within* the domain of numerical dependence.

The central idea of the Lagrangian scheme is to *represent the physical trajectory of the fluid parcel*.

We consider a parcel *arriving* at gridpoint $m\Delta x$ at the new time $(n+1)\Delta t$ and ask: Where has it come from?

The *departure point* will not normally be a grid point. Therefore, the value at the departure point must be calculated by *interpolation from surrounding points*.

But this interpolation ensures that the trajectory falls within the numerical domain of dependence.

We will show that this leads to a *numerically stable scheme*.

The advection equation in Lagrangian form may be written

$$\frac{dY}{dt} = 0\,.$$

In physical terms, this equation says that the value of Y is constant for a fluid parcel.

Applying the equation over the time interval $[n\Delta t, (n+1)\Delta t]$, we get

$$\begin{pmatrix}\text{Value of } Y \text{ at} \\ \text{point } m\Delta x \text{ at} \\ \text{time } (n+1)\Delta t\end{pmatrix} = \begin{pmatrix}\text{Value of } Y \text{ at} \\ \textit{departure point} \\ \text{at time } n\Delta t\end{pmatrix}$$

In a more compact form, we may write

$$Y^{n+1}_m = Y^n_\bullet$$

where $Y^n_\bullet$ represents the value at the departure point, which is normally not a grid point.

Interpolation using Surrounding Points

```
+--------+--------+--------+--------+--------o--------+ n+1
|        |        |        |        |   o    |        |
|        |        |        |       o|        |        |
+--------+--------+--------+++●++++++--------+--------+ n
m-5      m-4      m-3      m-2      m-1      m        m+1
```

The distance travelled in time Δt is $s = c\Delta t$.
The Courant Number is $\mu = \frac{c\Delta t}{\Delta x}$. Here, $\mu = \frac{5}{3}$. We define:

$$p = [\mu] = \text{ Integral part of } \mu$$
$$\alpha = \mu - p = \text{ Fractional part of } \mu$$

Note that, by definition, $0 \le \alpha < 1$ (here, $p = 1$ and $\alpha = 2/3$).
So, the departure point falls between the grid points $m - p - 1$ and $m - p$.
A *linear interpolation* gives

$$Y^n_\bullet = \alpha Y^n_{m-p-1} + (1-\alpha) Y^n_{m-p} .$$

Check: Show what this implies in the limits $\alpha = 0$ and $\alpha \to 1$.

Break here

Numerical Stability of the Scheme

The discrete equation may be written

$$Y^{n+1}_m = \alpha Y^n_{m-p-1} + (1-\alpha) Y^n_{m-p} .$$

Let us look for a solution of the form

$$Y^n_m = a\, A^n \exp(ikm\Delta x) .$$

Substituting into the equation we get

$$\begin{aligned} aA^{n+1}\exp(ikm\Delta x) = &\ \alpha \cdot aA^n \exp[ik(m-p-1)\Delta x] \\ + &\ (1-\alpha)\cdot aA^n \exp[ik(m-p)\Delta x] \end{aligned}$$

Removing the common term $aA^n \exp(ikm\Delta x)$, we get

$$A = \alpha \exp[ik(-p-1)\Delta x] + (1-\alpha)\exp[ik(-p)\Delta x]$$

We can write this as

$$A = \exp(-ikp\Delta x)\cdot[(1-\alpha) + \alpha\exp(-ik\Delta x)]$$

Again,

$$A = \exp(-ikp\Delta x)\cdot[(1-\alpha) + \alpha\exp(-ik\Delta x)]$$

Now consider the squared modulus of A:

$$\begin{aligned} |A|^2 &= |\exp(-ikp\Delta x)|^2 \cdot |(1-\alpha) + \alpha\exp(-ik\Delta x)|^2 \\ &= |(1-\alpha) + \alpha\cos k\Delta x - i\alpha\sin k\Delta x|^2 \\ &= [(1-\alpha) + \alpha\cos k\Delta x]^2 + \alpha[\sin k\Delta x]^2 \\ &= (1-\alpha)^2 + 2(1-\alpha)\alpha\cos k\Delta x + \alpha^2\cos^2 k\Delta x + \alpha^2\sin^2 k\Delta x \\ &= (1-2\alpha+\alpha^2) + 2\alpha(1-\alpha)\cos k\Delta x + \alpha^2 \\ &= 1 - 2\alpha(1-\alpha)[1-\cos k\Delta x] . \end{aligned}$$

We note that, for all θ, we have $0 \le (1-\cos\theta) \le 2$.

Taking the largest value of $1 - \cos k\Delta x$ gives

$$|A|^2 = 1 - 4\alpha(1-\alpha) = (1-2\alpha)^2 \le 1 .$$

Taking the smallest value of $1 - \cos k\Delta x$ gives

$$|A|^2 = 1 .$$

In either case, $|A|^2 \le 1$, so *there is numerical stability.*

Discussion and Conclusion

- ***We have determined the departure point by linear interpolation.***
- ***This ensures that $0 \leq \alpha < 1$.***
- ***This in turn ensures that $|A| \leq 1$.***
- ***In other words, we have unconditional numerical stability.***
- ***The implication is that the time step is unlimited.***
- ***In contradistinction to the Eulerian scheme there is no CFL criterion.***

- ***Of course, we must consider accuracy as well as stability***
- ***The time step Δt is chosen to ensure sufficient accuracy, but can be much larger than for an Eulerian scheme.***
- ***Typically, Δt is about six times larger for a semi-Lagrangian scheme than for an Eulerian scheme.***
- ***This is a substantial gain in computational efficiency.***

⋆ ⋆ ⋆

Miscellaneous Issues

- ***Calculation of departure point***
- ***Higher order interpolation***
- ***Interpolation in two dimensions***
- ***Interpolation in the vertical***
- ***Coriolis terms: Pseudo-implicit scheme***
- ***Inclusion of Physics***

End of §3.2.6