§3.2.5. Semi-implicit Schemes

The possibility of using a time step with a Courant number much larger than 1 in an implicit scheme does not guarantee that we will obtain accurate results economically.

The implicit scheme maintains stability by slowing down the solutions, so that the waves satisfy the CFL condition.

We saw this clearly in the analysis of the six-point Crank-Nicholson scheme.

For this reason, implicit schemes are useful for those modes that are very fast but of little meteorological importance.

We will next consider schemes in which the gravity wave terms are implicit while the remaining terms are explicit.

These semi-implicit schemes are of crucial importance in modern operational NWP.

Introduction to the SI Method

It is common practice today to treat selected linear terms implicitly and the remaining terms explicitly.

The semi-implicit method was pioneered by André Robert.

The terms that give rise to high frequency gravity waves are integrated implicitly, enabling the use of a long time step.

Formally, we separate the terms into two groups.
Thus, the equation

$$\frac{\partial u}{\partial t} = F(u) = F_1(u) + F_2(u)$$

is discretised by something like

$$\left(\frac{U^{n+1} - U^{n-1}}{2\Delta t}\right) = F_1(U^n) + F_2\left(\frac{U^{n-1} + U^{n+1}}{2}\right)$$

Schemes of this sort are pivotal in modern NWP models, due to their excellent stability properties.

Semi-implicit Scheme for SWE

We consider the Shallow Water Equations:

$$\left.\begin{array}{l}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\underline{\frac{\partial \Phi}{\partial x}} + fv \\[2mm] \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\underline{\frac{\partial \Phi}{\partial y}} - fu \\[2mm] \frac{\partial \Phi}{\partial t} + u\frac{\partial \Phi}{\partial x} + v\frac{\partial \Phi}{\partial y} = -\underline{\bar{\Phi}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)} - \left(\Phi - \bar{\Phi}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\end{array}\right\}$$

The terms that lead to fast gravity waves are underlined.

The phase speed of the inertia-gravity wave is given by

$$c = \bar{U} \pm \sqrt{\bar{\Phi} + \frac{f^2}{k^2}} \approx \bar{U} \pm 300 \,\text{m/s}$$

The Courant number $\mu = c\Delta t/\Delta x$ is dominated by the speed of external inertia-gravity waves, $c = c_{IGW}$.
An *explicit scheme* thus requires a time step an order of magnitude smaller than that required for advection.

Robert (1969) introduced the use of semi-implicit schemes to slow down the gravity waves.

We will develop the semi-implicit scheme first for continuous variation in space, and then for discretised space.

The shallow water equations may be written

$$\left.\begin{array}{l}\frac{\partial u}{\partial t} = -\underline{\frac{\partial \Phi}{\partial x}} + R_u \\[2mm] \frac{\partial v}{\partial t} = -\underline{\frac{\partial \Phi}{\partial y}} + R_v \\[2mm] \frac{\partial \Phi}{\partial t} = -\underline{\bar{\Phi}\delta} + R_\Phi\end{array}\right\}$$

We discretise implicitly the terms that result in gravity waves, and explicitly the remaining terms.

$$\left(\frac{u^{n+1} - u^{n-1}}{2\Delta t}\right) = -\left(\frac{\Phi_x^{n+1} + \Phi_x^{n-1}}{2}\right) + R_u^n$$

$$\left(\frac{v^{n+1} - v^{n-1}}{2\Delta t}\right) = -\left(\frac{\Phi_y^{n+1} + \Phi_y^{n-1}}{2}\right) + R_v^n$$

$$\left(\frac{\Phi^{n+1} - \Phi^{n-1}}{2\Delta t}\right) = -\bar{\Phi}\left(\frac{\delta^{n+1} + \delta^{n-1}}{2}\right) + R_\Phi^n$$

Now solve for u^{n+1} and v^{n+1}:

$$u^{n+1} = -\Delta t\,\Phi_x^{n+1} + S_u$$
$$v^{n+1} = -\Delta t\,\Phi_y^{n+1} + S_v$$

Then, the divergence at time $(n+1)\Delta t$ is

$$\delta^{n+1} = (u_x^{n+1} + v_y^{n+1})$$
$$= -\Delta t(\Phi_{xx}^{n+1} + \Phi_{yy}^{n+1}) + R_\delta$$
$$= -\Delta t\nabla^2\Phi^{n+1} + R_\delta$$

We substitute this in the continuity equation to get

$$\left(\frac{\Phi^{n+1} - \Phi^{n-1}}{2\Delta t}\right) = \frac{1}{2}\bar{\Phi}\Delta t\,\nabla^2\Phi^{n+1} + S_\Phi$$

This can be written as a **Helmholtz Equation** for Φ^{n+1}:

$$\left[\nabla^2 - \left(\frac{1}{\bar{\Phi}\Delta t^2}\right)\right]\Phi^{n+1} = F_\Phi$$

Once we solve this for Φ^{n+1}, the velocity components are obtained from

$$u^{n+1} = -\Delta t\,\Phi_x^{n+1} + S_u$$
$$v^{n+1} = -\Delta t\,\Phi_y^{n+1} + S_v$$

All the variables are now known at time $(n+1)\Delta t$ and the next time-step can be computed.

Spatial Discretisation

We introduce compact notation for finite differences and averages:

$$\delta_x f = \frac{f_{j+1/2} - f_{j-1/2}}{\Delta x} \qquad \text{Space difference}$$

$$\overline{f}^x = \tfrac{1}{2}(f_{j+1/2} + f_{j-1/2}) \qquad \text{Space average}$$

and similarly for differences in y or t.

With this notation, assuming uniform resolution, we have

$$\delta_{2x} f = \delta_x \overline{f}^x = \frac{f_{i+1} - f_{i-1}}{2\Delta x}$$

$$\overline{f}^{2x} = \tfrac{1}{2}(f_{i+1} + f_{i-1})$$

and, for time,

$$\overline{f}^{2t} = \frac{1}{2}(f^{n+1} + f^{n-1})$$

Using this compact finite difference notation, we can write the leapfrog semi-implicit SWE as

$$\delta_{2t}u + u\,\delta_{2x}u + v\,\delta_{2y}u = -\delta_{2x}\overline{\Phi}^{2t} + fv$$
$$\delta_{2t}v + u\,\delta_{2x}v + v\,\delta_{2y}v = -\delta_{2y}\overline{\Phi}^{2t} - fu$$
$$\delta_{2t}\Phi + u\,\delta_{2x}\Phi + v\,\delta_{2y}\Phi = -\bar{\Phi}\overline{(\delta_{2x}u + \delta_{2y}v)}^{2t} - (\Phi - \bar{\Phi})(\delta_{2x}u + \delta_{2y}v)$$

Everything that **does not have a time average** involves only terms evaluated explicitly at the nth time step.

We can rearrange the FDEs as

$$\frac{u^{n+1} - u^{n-1}}{2\Delta t} = -\delta_{2x}\left(\frac{\Phi^{n+1} + \Phi^{n-1}}{2}\right) + R_u$$

$$\frac{v^{n+1} - v^{n-1}}{2\Delta t} = -\delta_{2y}\left(\frac{\Phi^{n+1} + \Phi^{n-1}}{2}\right) + R_v$$

$$\frac{\Phi^{n+1} - \Phi^{n-1}}{2\Delta t} = -\bar{\Phi}\left[\delta_{2x}\left(\frac{u^{n+1} + u^{n-1}}{2}\right) + \delta_{2y}\left(\frac{v^{n+1} + v^{n-1}}{2}\right)\right] + R_\Phi$$

The terms R_u, R_v and R_Φ are the remaining terms, evaluated at the central time $n\Delta t$:

$$R_u = -u\delta_{2x}u - v\delta_{2y}u + fv$$
$$R_v = -u\delta_{2x}v - v\delta_{2y}v - fu$$
$$R_\Phi = -u\delta_{2x}\Phi - v\delta_{2y}\Phi - \left(\Phi - \bar{\Phi}\right)(\delta_{2x}u + \delta_{2y}v)$$

We can solve the first two equations for u^{n+1} and v^{n+1}:

$$\left.\begin{array}{c} u^{n+1} = -\Delta t\delta_{2x}\Phi^{n+1} + S_u \\[2mm] v^{n+1} = -\Delta t\delta_{2y}\Phi^{n+1} + S_v \end{array}\right\}$$

where S_u and S_v can be computed from known quantities.

Eliminating u^{n+1} and v^{n+1} from the third equation, we obtain an elliptic equation for Φ^{n+1}:

$$\left(\delta_{2x}^2 + \delta_{2y}^2 - \frac{1}{\bar{\Phi}\Delta t^2}\right)\Phi^{n+1} = F_{i,j}^n$$

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Again,

$$\left(\delta_{2x}^2 + \delta_{2y}^2 - \frac{1}{\bar{\Phi}\Delta t^2}\right)\Phi^{n+1} = F_{i,j}^n$$

The right-hand side of this **Helmholtz equation** depends only on values at $t = n\Delta t$ or $(n-1)\Delta t$, so that it is known.

Solving this elliptic equation provides Φ^{n+1}.

Once this is known, it can be plugged back into the first two equations. Thus (u^{n+1}, v^{n+1}) can be obtained.

The elliptic operator on the left-hand side of the Helmholtz equation is a finite difference equivalent to $(\nabla^2 - \lambda^2)$,

$$\left(\delta_{2x}^2 + \delta_{2y}^2 - \frac{1}{\bar{\Phi}\Delta t^2}\right)\Phi = \frac{\Phi_{i+2,j} + \Phi_{i-2,j} + \Phi_{i,j+2} + \Phi_{i,j-2} - \left(4 + \frac{1}{\mu^2}\right)\Phi_{i,j}}{4\Delta^2}$$

We assume for simplicity that $\Delta x = \Delta y = \Delta$.

Here, $\mu^2 = \bar{\Phi}\Delta t^2/\Delta^2$ is the Courant number (squared) for gravity waves.

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Since $\mu^2 = \bar{\Phi}\Delta t^2/\Delta^2 >> 1$, the semi-implicit scheme distorts the gravity wave solution, slowing the gravity wave down until they satisfy the CFL criterion.

This is an acceptable distortion since we are interested in the slower "weather-like" processes.

Since the slower modes are written explicitly, they are not slowed down or distorted in a significant way.

In a nut-shell:

- The fast gravity-waves are slowed down by the semi-implicit scheme. Thus, their behaviour is distorted.
- The slower, meteorologically significant Rossby-Haurwitz waves are represented accurately, as long as the Courant Number for these waves is not too large.

In the same way the three-dimensional divergence in the continuity equation, the term giving rise to sound waves, can also be written semi-implicitly.

This change has allowed the use of non-hydrostatic models without the use of the anelastic approximation or the hydrostatic approximation.

André Robert (1982) created a model that Eugenia Kalnay describes as the ultimate atmospheric model.

It treats the terms generating sound waves (anelastic terms, i.e., three-dimensional divergence), and the terms generating gravity waves (pressure gradient and horizontal divergence) semi-implicitly,

It uses a 3-D semi-Lagrangian scheme for the advection.

This model, called the Mesoscale Compressible Community (MCC) model, is a *universal model*, designed to tackle accurately atmospheric problems from the planetary scale through mesoscale, convective and smaller.

Google for MCC model

See documentation of LM model

See documentation of ECMWF model

Conclusion of §3.2.5