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**We can show that this is stable provided**

$$(2\kappa\Delta t + \omega^2 \Delta t^2) \leq 1 \, .$$

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**The Navier-Stokes equations are**

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**NWP models also use various filtering processes to limit spatial and temporal noise.**

**Some of these represent diffusive physical processes. Others are just numerical damping, to prevent spurious noise.**

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**In the two-dimensional case, the stability criterion is more stringent: we need to choose a time step that is $\sqrt{2}$ times smaller than that permitted in the one-dimensional case.**

# Implicit Schemes

**For the simple oscillation equation**

$$\frac{dU}{dt} = i\omega U$$

**the (centered) implicit approximation is**

$$\frac{U^{n+1} - U^n}{\Delta t} = i\omega \left( \frac{U^{n+1} + U^n}{2} \right) .$$

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**This is second-order accurate and unconditionally stable:**

$$U^{n+1} = \rho U^n \qquad \textbf{where} \qquad \rho = \left( \frac{1 + \frac{1}{2} i\omega\Delta t}{1 - \frac{1}{2} i\omega\Delta t} \right)$$

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**Exercise: Verify that $\rho$ is unimodular: $|\rho| = 1$.**

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**Formally, we separate the terms into two groups.**

**Thus, the equation**

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**Schemes of this sort are pivotal in modern NWP models, due to their excellent stability properties.**

# Distortion of the Phase Speed

**We consider the simple 1-D advection equation**

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**We use centered difference approximations**

$$\left(\frac{U_m^{n+1} - U_m^{n-1}}{2\Delta t}\right) + c\left(\frac{U_{m+1}^n - U_{m-1}^n}{2\Delta x}\right) = 0\,,$$

**in both time and space (CTCS). Here** $U_m^n = U(m\Delta x, n\Delta t)$**.**

**Again, the CTCS or leapfrog scheme is**

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**Substituting $U_m^n$ into the FDE, we find that**

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**Exercise: Verify this expression for $C$.**

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**Thus, the condition for stability of the solution is**

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$$Le \equiv \frac{c\Delta t}{\Delta x} \leq 1 .$$

**This non-dimensional parameter is often called the Courant number, but is denoted here as $Le$ for Lewy, who first discovered this stability criterion.**

**The above analysis may be repeated for an implicit discretization (six-point Crank-Nicholson scheme):**

$$\frac{U_m^{n+1} - U_m^n}{\Delta t} + \frac{c}{2}\left(\frac{U_{m+1}^n - U_{m-1}^n}{2\Delta x} + \frac{U_{m+1}^{n+1} - U_{m-1}^{n+1}}{2\Delta x}\right) = 0\,.$$

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**Then the phase speed $C$ of the numerical solution is**

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**Exercise: Verify this result. Hint: Substitute $U_m^n = U^0 \exp[ik(m\Delta x - Cn\Delta t)]$ into the equation.**

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**This equation contains an inverse tangent term instead of the inverse sine occurring in the leapfrog scheme.**

**Thus, the numerical phase speed $C$ is always real, so the scheme is *unconditionally stable*.**

**It is easily shown that $C \leq c$ and that $C \to \pi/k\Delta t$ as $c \to \infty$.**

**Thus, the implicit scheme slows down the faster waves.**

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## MatLab Exercise:

- **Write a program to evaluate**

$$C = \frac{1}{k\Delta t}\sin^{-1}\left[\left(\frac{c\Delta t}{\Delta x}\right)\sin k\Delta x\right] .$$

  **and determine the behaviour of $C$ in the limits $c = 0$ and $c \longrightarrow \infty$.**

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**Then the relationships can be written:**

- $\dfrac{C}{c} = \dfrac{1}{\kappa\mu}\sin^{-1}(\mu\sin\kappa)$ **for the explicit scheme.**
- $\dfrac{C}{c} = \dfrac{1}{\kappa\mu}\tan^{-1}(\frac{1}{2}\mu\sin\kappa)$ **for the implicit scheme.**

**Now there are only two parameters, $\kappa$ and $\mu$.**

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**You should plot curves of $C/c$ as functions of $\mu$ for a selection of values of $\kappa$, say for $\kappa \in \{0, \frac{\pi}{10}, \frac{2\pi}{10}, \ldots, \pi\}$, with $\mu$ varying from zero to, say, 10.**

# Exercise.

**Consider the four-point Crank-Nicholson scheme**

$$\frac{1}{2}\left[\frac{U_m^{n+1}-U_m^n}{\Delta t}+\frac{U_{m+1}^{n+1}-U_{m+1}^n}{\Delta t}\right]+\frac{c}{2}\left[\frac{U_{m+1}^{n+1}-U_m^{n+1}}{\Delta x}+\frac{U_{m+1}^n-U_m^n}{\Delta x}\right]=0$$

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# Implicit Time Schemes

**In implicit schemes the advection or diffusion terms are written in terms of the new time level variables.**

$$\textbf{PDE:} \quad \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

$$\textbf{FDE:} \quad \frac{1}{2}\left[\left(\frac{U_m^{n+1} - U_m^n}{\Delta t}\right) + \left(\frac{U_{m+1}^{n+1} - U_{m+1}^n}{\Delta t}\right)\right] + c\left[\alpha\left(\frac{U_{m+1}^n - U_m^n}{\Delta x}\right) + (1-\alpha)\left(\frac{U_{m+1}^{n+1} - U_m^{n+1}}{\Delta x}\right)\right] = 0$$

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**The factor $\alpha$ determines the weight of the "old" time values compared with the "new" time values in the FDE.**

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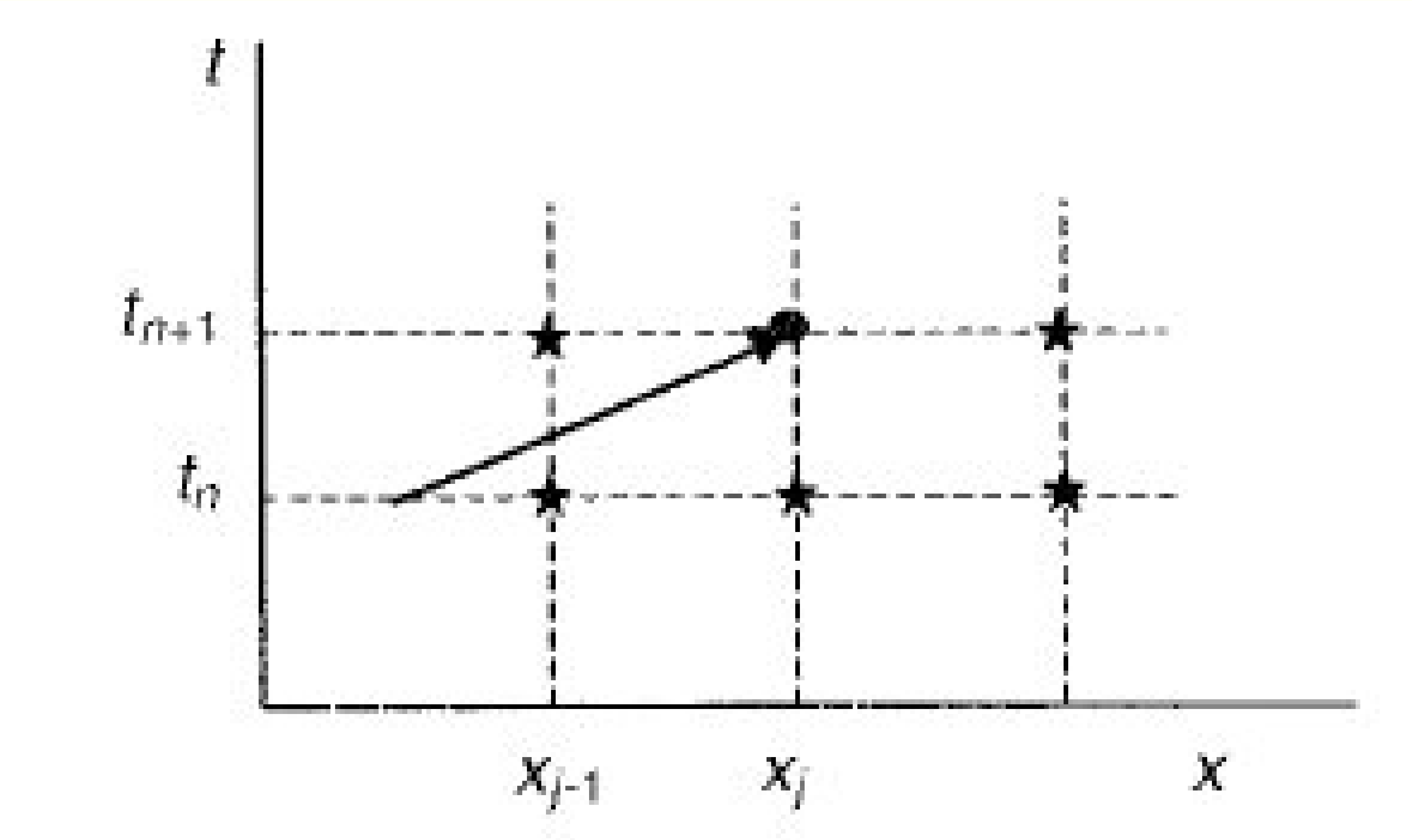
**This property is useful for solving problems such as spuriously growing mountain waves in semi-Lagrangian schemes.**




**Schematic of an implicit scheme. Note that with the implicit scheme there is no extrapolation.**

# Summary I

**If we consider a marching equation**

$$\frac{dU}{dt} = F(U)$$

**explicit methods such as the forward scheme**

$$\frac{U^{n+1} - U^n}{\Delta t} = F(U^n)$$

**or the leapfrog scheme**

$$\frac{U^{n+1} - U^{n-1}}{2\Delta t} = F(U^n)$$

**are either**

- **Conditionally stable, or**
- **Absolutely unstable.**

# Summary II

**A fully implicit scheme**

$$\frac{U^{n+1} - U^n}{\Delta t} = F(U^{n+1})$$

**and a centered implicit scheme**

$$\frac{U^{n+1} - U^n}{\Delta t} = F\left(\frac{U^n + U^{n+1}}{2}\right)$$

**are absolutely stable.**

**The latter scheme is attractive because it is centered in time, and it can be written with centered space differences, which makes it second order in space and in time.**

**As these schemes have only two time levels,**
**they have no computational mode.**

**Break here**

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- **The four-point C-N scheme:**

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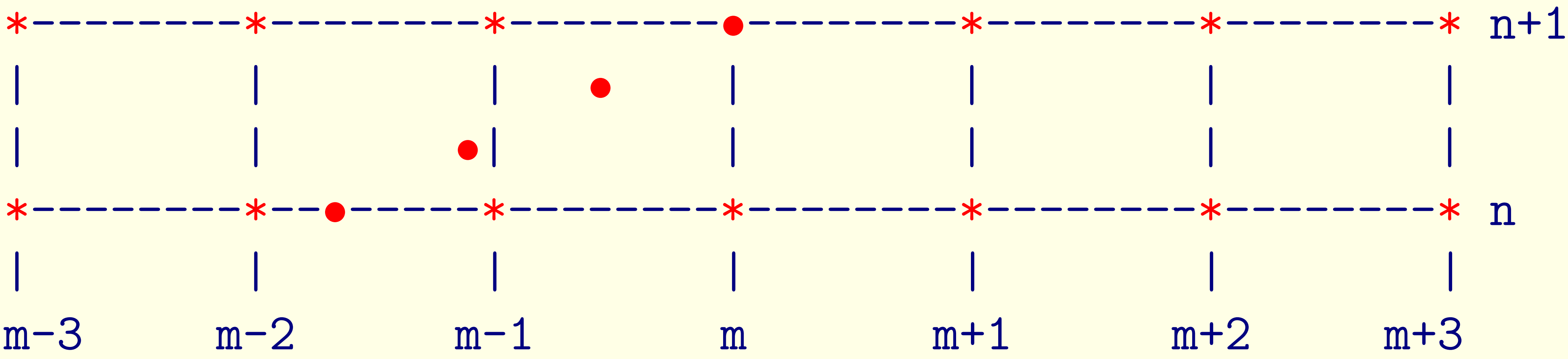
- **The six-point C-N scheme:**

$$\frac{U_m^{n+1}-U_m^n}{\Delta t}+\frac{c}{2}\left[\frac{U_{m+1}^{n+1}-U_{m-1}^{n+1}}{2\Delta x}+\frac{U_{m+1}^n-U_{m-1}^n}{2\Delta x}\right]=0$$

# Domain of Dependence of Implicit Scheme



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```
*---------*---------*---------●---------*---------*---------* n+1
|         |         |    ●    |         |         |         |
|         |        ●|         |         |         |         |
*---------*--●------*---------*---------*---------*---------* n
|         |         |         |         |         |         |
m-3      m-2       m-1        m        m+1       m+2       m+3
```

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**The value at the point $m\Delta x$ at time $(n+1)\Delta t$ depends on all the points denoted by red asterisks (*).**

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**Thus, the computational domain of dependence surrounds the physical domain of dependence.**

**This is a necessary condition for a stable scheme.**

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**These so-called ADI (alternating direction implicit) schemes allow large time steps without a large additional computational cost.**

# Example of a Linear System.

**The 1-D advection equation on a periodic domain is**

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**We can write this in matrix form (with $\mu = c\Delta t/4\Delta x$)**

$$\begin{bmatrix} 1 & +\mu & 0 & \dots & -\mu \\ -\mu & 1 & +\mu & \dots & 0 \\ 0 & -\mu & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ +\mu & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} U_0^{n+1} \\ U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_{M-1}^{n+1} \end{bmatrix} = \begin{bmatrix} 1 & -\mu & 0 & \dots & +\mu \\ +\mu & 1 & -\mu & \dots & 0 \\ 0 & +\mu & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\mu & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} U_0^{n} \\ U_1^{n} \\ U_2^{n} \\ \vdots \\ U_{M-1}^{n} \end{bmatrix}$$

**where $x_M = M\Delta x$ and $U_M^n = U_0^n$ for all $n$.**

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**If the non-linear terms are treated implicitly, we must solve a nonlinear algebraic system every time step.**

**This is normally *impractical*.**

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**Exercise: See Notes and Exercises, Kalnay, pp. 87–88.**

## Conclusion of §3.2.4