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Thus, the continuous evolution of the variables is approximated by the change from step to step.

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Lewis Fry Richardson described the procedure:

Although the infinitesimal calculus has been a splendid success, yet there remain problems in which it is cumbrous or unworkable. When such difficulties are encountered it may be well to return to the manner in which they did things before the calculus was invented, postponing the passage to the limit until after the problem has been solved for a moderate number of moderately small differences.

(Richardson, 1927)

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The solution at these moments is denoted by $U^n = U(n\Delta t)$.

If this solution is known up to time $t = n\Delta t$, the right-hand term $F^n = F(U^n)$ can be computed.

Thus, we can integrate the equation forward in time.

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This process of stepping forward from moment to moment is repeated a large number of times, until the desired forecast range is reached.

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The computational initial condition can be defined in several ways:

- **Set** $U^1 = U^0$**. Since** $u^1 = u^0 + u_t\Delta t + \cdots$**, this introduces errors of order** $O(\Delta t)$**, and is not recommended.**

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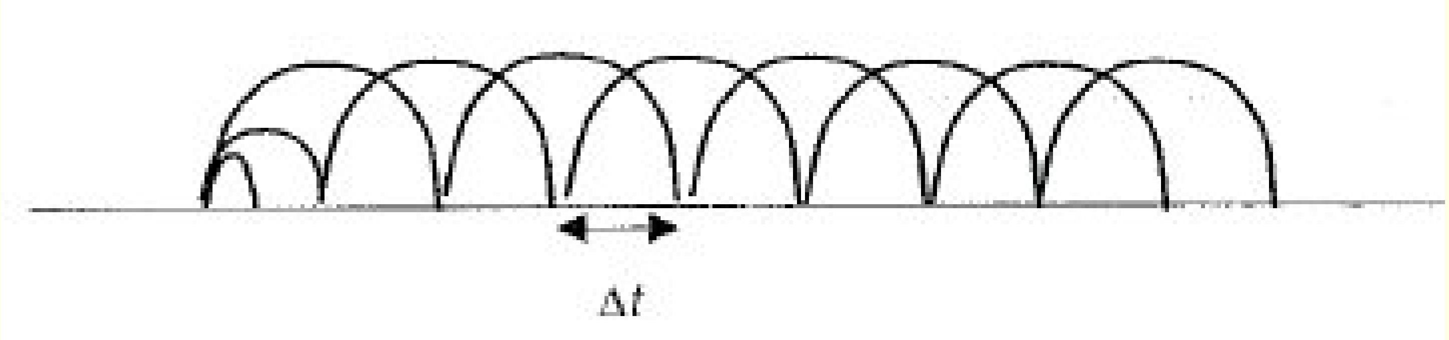
- **Use half of the initial time step for the forward time step, followed by leapfrog time steps. This will reduce the error introduced in the unstable first step.**



Schematic of the leapfrog scheme with a small starting step.

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We analyse the oscillation equation with $\omega = 0$:

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This illustrates the importance of a careful choice of the computational initial condition.

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Note that the added term is like smoothing in time, an approximation of an ideally time-centered smoother:

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The smoother

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reduces the amplitude of different frequencies ν **by a factor** $(1 - 4\gamma \sin^2(\nu \Delta t/2))$**.**

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This filter is widely used with the leapfrog scheme, with γ of the order of 0.01.

Time schemes for $dU/dt = F(U)$

(a) $\frac{U^{n+1}-U^{n-1}}{2\Delta t} = F(U^n)$	Leapfrog (good for hyperbolic equations, unstable for parabolic equations)
(a′) $\frac{U^{n+1}-\overline{U}^{n-1}}{2\Delta t} = F(U^n)$; $\overline{U}^n = U^n + \alpha(U^{n+1} - 2U^n + \overline{U}^{n-1})$	Leapfrog smoothed with the Robert–Asselin time filter; $\alpha \sim 1\%$
(b) $\frac{U^{n+1}-U^n}{\Delta t} = F(U^n)$	Euler (forward, good for diffusive terms, unstable for hyperbolic equations)
(c) $\frac{U^{n+1}-U^n}{\Delta t} = F\left(\frac{U^n+U^{n+1}}{2}\right)$	Crank–Nicholson or centered implicit
(c′) $\frac{U^{n+1}-U^n}{\Delta t} = F\left(\frac{\beta U^n+(1-\beta)U^{n+1}}{2}\right)$; $\beta < 0.5$	Implicit, slightly damping
(d) $\frac{U^{n+1}-U^n}{\Delta t} = F(U^{n+1})$	Fully implicit or backward

Time schemes for $dU/dt = F(U)$

(e) $\frac{U^* - U^n}{\Delta t} = F(U^n)$; $\frac{U^{n+1} - U^n}{\Delta t} = F(U^*)$

Euler-backward or Matsuno: good for damping high frequency waves

(f) $\frac{U^* - U^n}{\Delta t} = F(U^n)$;

$\frac{U^{n+1} - U^n}{\Delta t} = F\left(\frac{U^n + U^*}{2}\right)$

Another predictor–corrector scheme (Heun)

(g) $\frac{U^{n+1} - U^n}{\Delta t} = F\left(\frac{3}{2}U^n - \frac{1}{2}U^{n-1}\right)$

Adams–Bashford (second order in time).

(h) $\frac{U^{n+1/2^*} - U^n}{\Delta t/2} = F(U^n)$;

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$$\frac{U^{n+1} - U^n}{\Delta t} = \frac{1}{6}[F(U^n) + 2F(U^{n+1/2^*}) + 2F(U^{n+1/2^{**}}) + F(U^{n+1^*})]$$

Runge–Kutta (fourth order)

Time schemes for $dU/dt = F(U)$

(i)	$a = 0;\ b = 1/\Delta t$ $U^* \leftarrow (aU^* + F(U^n))/b$ $U^n \leftarrow U^n + U^*$ $a \leftarrow a - 1/(N\Delta t);\ b \leftarrow b - 1/(N\Delta t)$	N-times	Lorenz's N-cycle, $N =$ multiple of 4; Nth order
(j)	$\frac{U^{n+1} - U^{n-1}}{2\Delta t} = F_1(U^n) + F_2\left(\frac{U^{n+1} + U^{n-1}}{2}\right)$		Semi-implicit
(k)	$\frac{U^* - U^n}{\Delta t} = F_1(U^n);\ \frac{U^{n+1} - U^*}{\Delta t} = F_2(U^*)$		Fractional steps

For schemes (j) and (k), the right hand side is split into two terms: $F(U) = F_1(U) + F_2(U)$.

Break here

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Thus the solution U has a constant modulus, and a phase that increases or decreases linearly with time.

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We might attempt to obtain a more accurate solution by using a centered difference for the time derivative, as in the leapfrog scheme.

Let us look at this possibility now.

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Substituting this solution into the FDE, there are two possibilities:

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The leapfrog scheme for the Friction Equation is:

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Exercise: **Prove this. Hint: if** $y = -x + \sqrt{1 + x^2}$**, then** $y(0) = 1$**;** $y > 0$**;** $y' < 0$ **so** $0 < y \leq 1$**.**

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But we know better: read on!

The Oscillation Equation

If we apply the leapfrog scheme to the *oscillation equation*

$$\frac{dU}{dt} = i\omega U\,, \qquad \textbf{with} \quad U = U^0 \quad \textbf{at} \quad t = 0\,,$$

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Using the leapfrog scheme and again seeking a solution $U^n = U^0\rho^n$ for constant ρ, there are again two possibilities:

$$\rho_{\pm} = i\omega\Delta t \pm \sqrt{1 - \omega^2\Delta t^2}\,.$$

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	Friction Equation	Oscillation Equation
Euler Scheme	Conditionally Stable	UNSTABLE
Leapfrog Scheme	UNSTABLE	Conditionally Stable

Matlab Exercises

- **Write a MATLAB program to solve the oscillation equation**

$$\frac{dU}{dt} = iU\,, \quad U^0 = 1 \qquad\qquad (\omega = 1)$$

the analytical solution of which is $U(t) = \exp(it)$**, using**

– **the Euler forward method**

– **the leapfrog method**

Draw conclusions about the stability of the two schemes.

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- **Write a MATLAB program to solve the friction equation**

$$\frac{dU}{dt} = -U\,, \quad U^0 = 1 \qquad\qquad (\kappa = 1)$$

the analytical solution of which is $U(t) = \exp(-t)$**, using**

- **the Euler forward method**
- **the leapfrog method**

Draw conclusions about the stability of the two schemes.

Conclusion of §3.2.3