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  which is a hyperbolic equation.

- **The diffusion equation,**

$$\frac{\partial u}{\partial t} = \sigma\frac{\partial^2 u}{\partial x^2}$$

  which is a parabolic equation.

The Finite Difference Method

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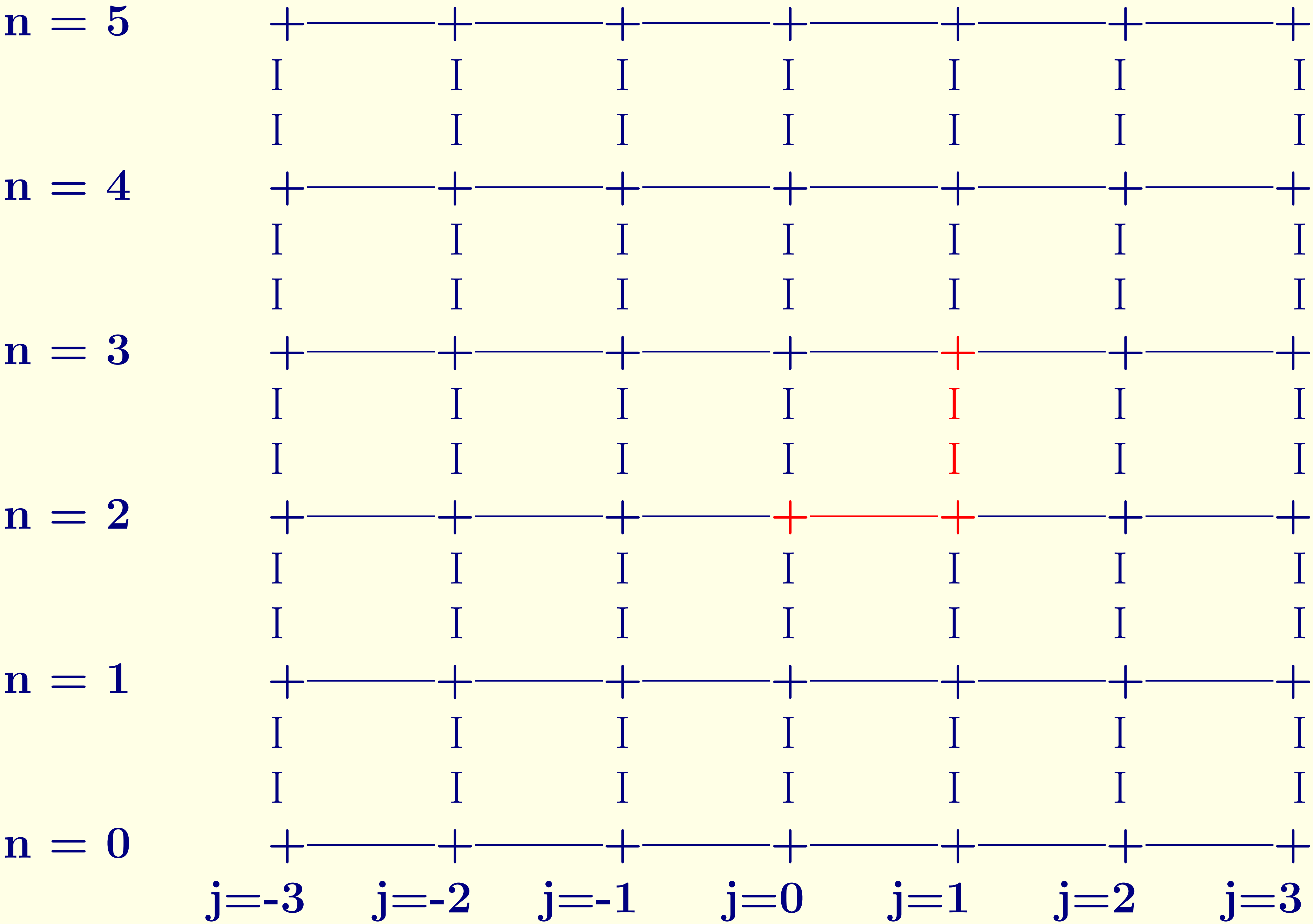
Suppose we choose to approximate this PDE with the FDE

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + c\,\frac{U_j^n - U_{j-1}^n}{\Delta x} = 0$$

Space-Time Grid:

Space axis horizontal
Time axis vertical



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- **[1] Is the FDE consistent with the PDE?**
- **[2] For a given time $t > 0$, will the solution of the FDE converge to that of the PDE as $\Delta x \to 0$ and $\Delta t \to 0$?**

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Warning: Sometimes superscript n denotes a *power*; sometimes it is just an index. Be careful!

Truncation Errors and Consistency

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- **Subtract the PDE from the FDE.**

If the difference (local truncation error) goes to zero as $\Delta x \to 0, \Delta t \to 0$, then the FDE is consistent with the PDE.

Example

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$$\left.\begin{aligned} u_j^{n+1} &= \left(u + u_t\Delta t + \tfrac{1}{2}u_{tt}\Delta t^2 + \cdots\right)_j^n \\ u_{j-1}^{n} &= \left(u - u_x\Delta x + \tfrac{1}{2}u_{xx}\Delta x^2 - \cdots\right)_j^n \end{aligned}\right\}$$

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$$\left(u_t + \tfrac{1}{2}u_{tt}\Delta t + \cdots\right)_j^n + c\left(u_x - \tfrac{1}{2}u_{xx}\Delta x + \cdots\right)_j^n \simeq 0$$

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Subtracting the PDE gives the local truncation error:

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Truncation errors are a crucial factor in determining forecast accuracy in NWP.

Convergence and Stability

The second question posed above was whether the solution of the FDE converges to the PDE solution.

That is, if we let $\Delta x \to 0$ and $\Delta t \to 0$, so that $j\Delta x \to x$ and $n\Delta t \to t$, does $U(j\Delta x, n\Delta t) \to u(x,t)$?

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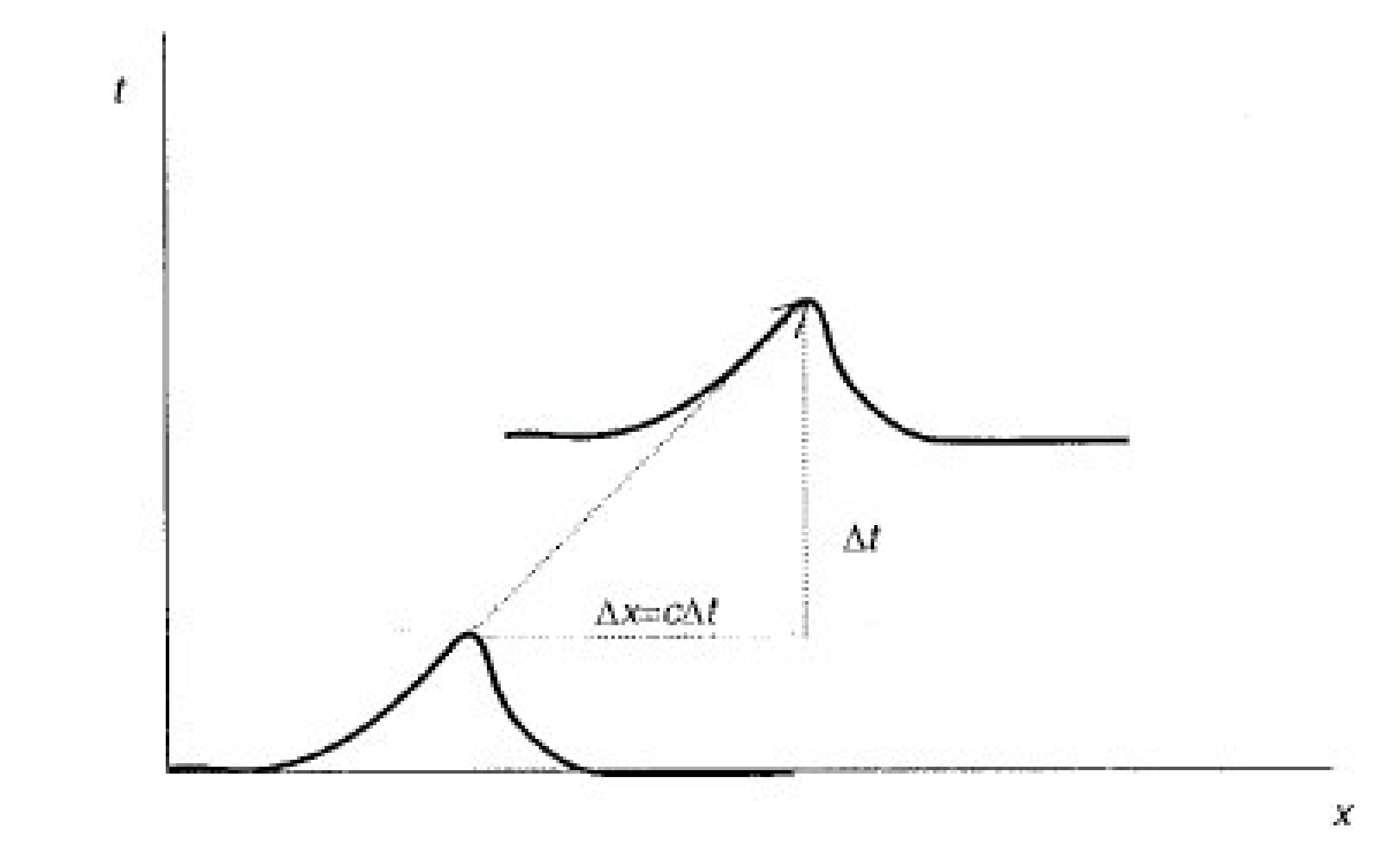
The shape of the solution $u(x,0)$ translates along the x-axis with velocity c (see Figure below).



Schematic of the solution of the advection equation (for $c > 0$).

The FDE for the upstream scheme can be written as

$$U_j^{n+1} = (1-\mu)U_j^n + \mu U_{j-1}^n$$

where

$$\mu \equiv \frac{c\Delta t}{\Delta x}$$

is the *Courant number* (or *Lewy Number*).

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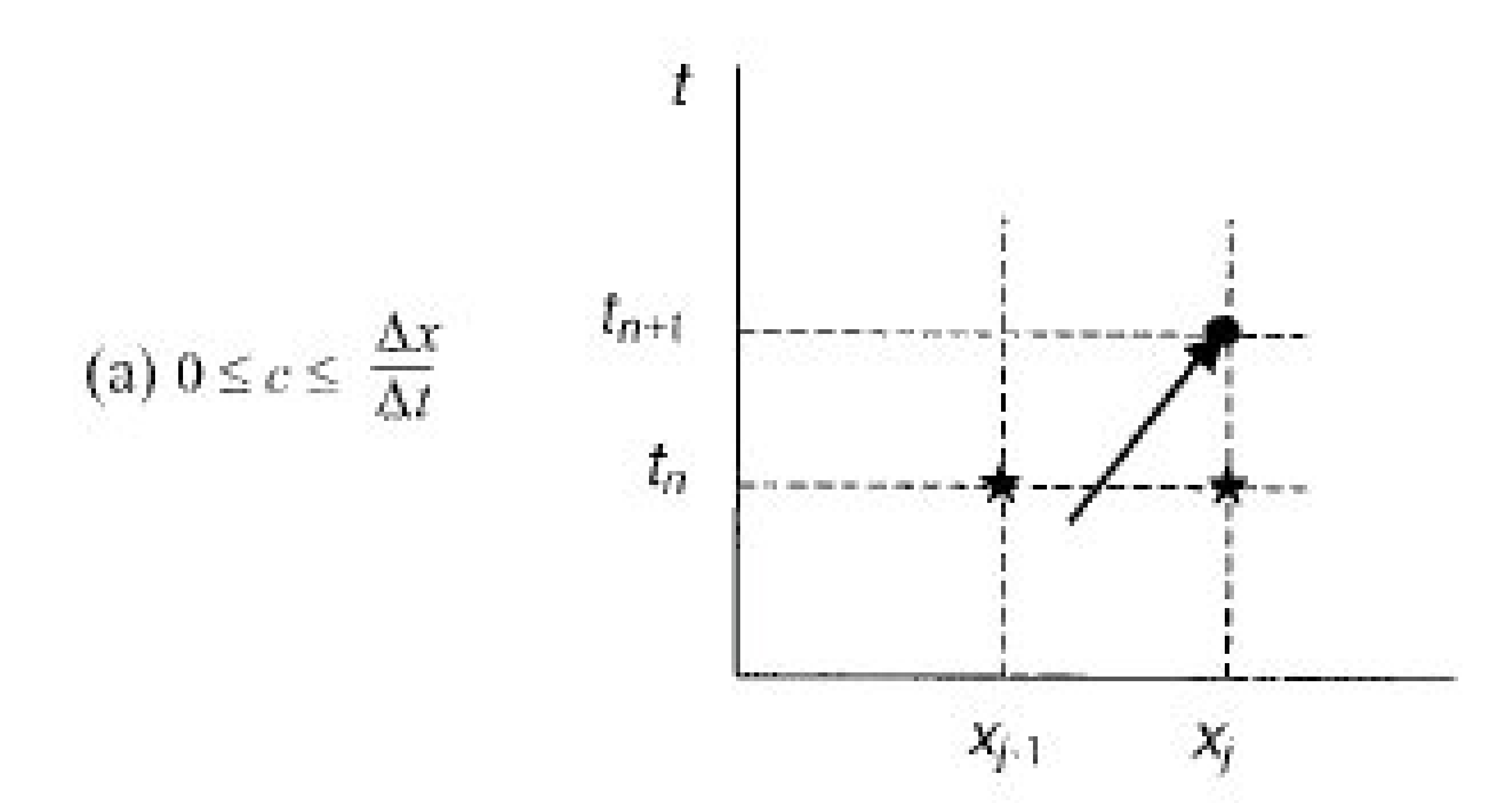
In this case the advection scheme works the way it should, because the true solution lies in between those values.



Schematic of the relationship between Δx, Δt and c leading to interpolation of the solution at time-level $n + 1$.

$$0 < \mu \equiv \frac{c\Delta t}{\Delta x} < 1$$

However, suspect this condition is not satisfied, so that

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Then the parcel arriving at point x_j at time t_{n+1} comes from somewhere outside the interval (x_{j-1}, x_j) at time t_n.

[Recall that $\partial u/\partial t + c\partial u/\partial x = 0$ is a linear approximation to $du/dt = 0$.]

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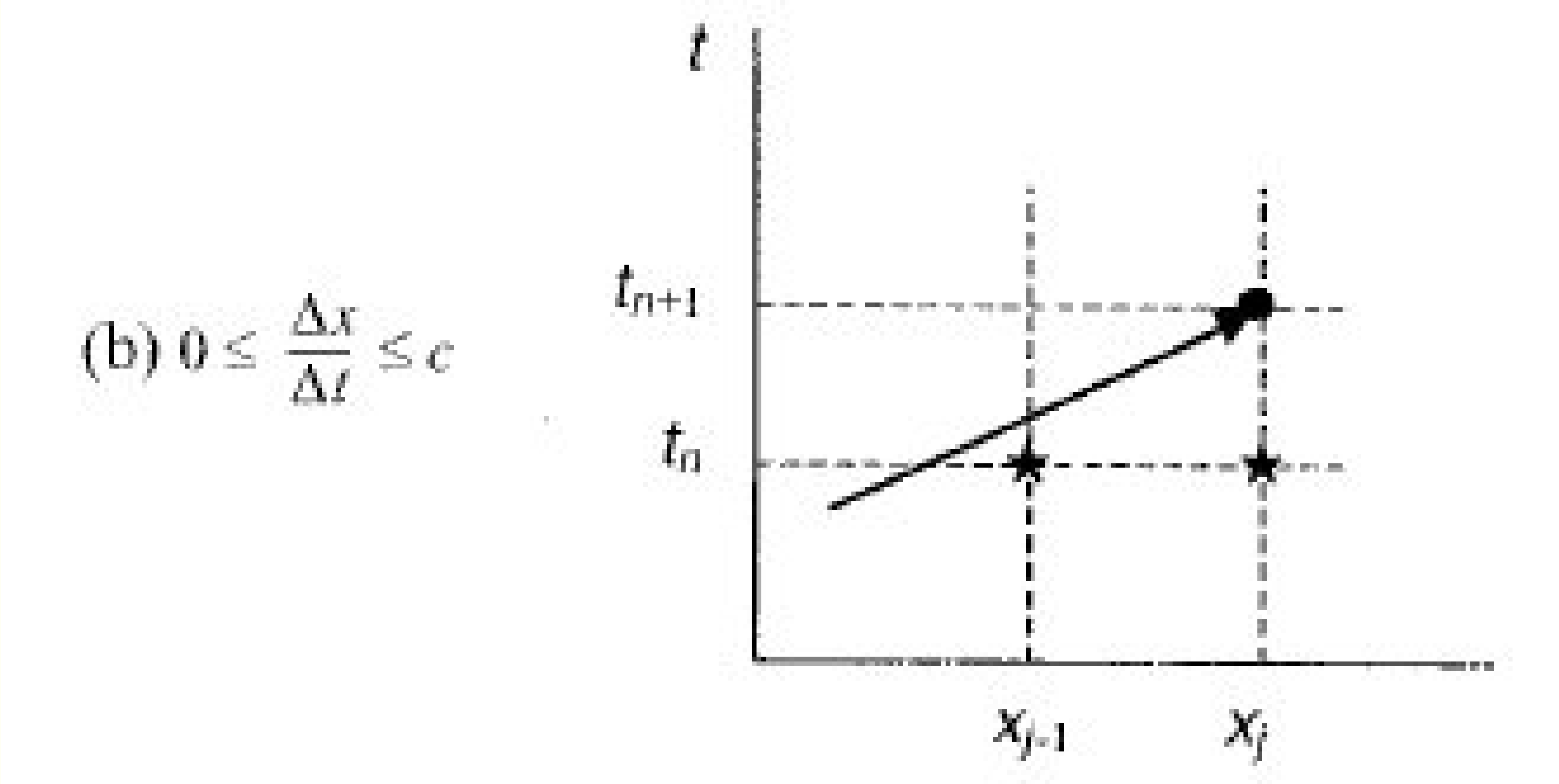
[**Recall that $\partial u/\partial t + c\partial u/\partial x = 0$ is a linear approximation to $du/dt = 0$.**]

Thus, the value of $\mathbf{U}_j^{n+1}$ is extrapolated from the values U_j^n and U_{j-1}^n.



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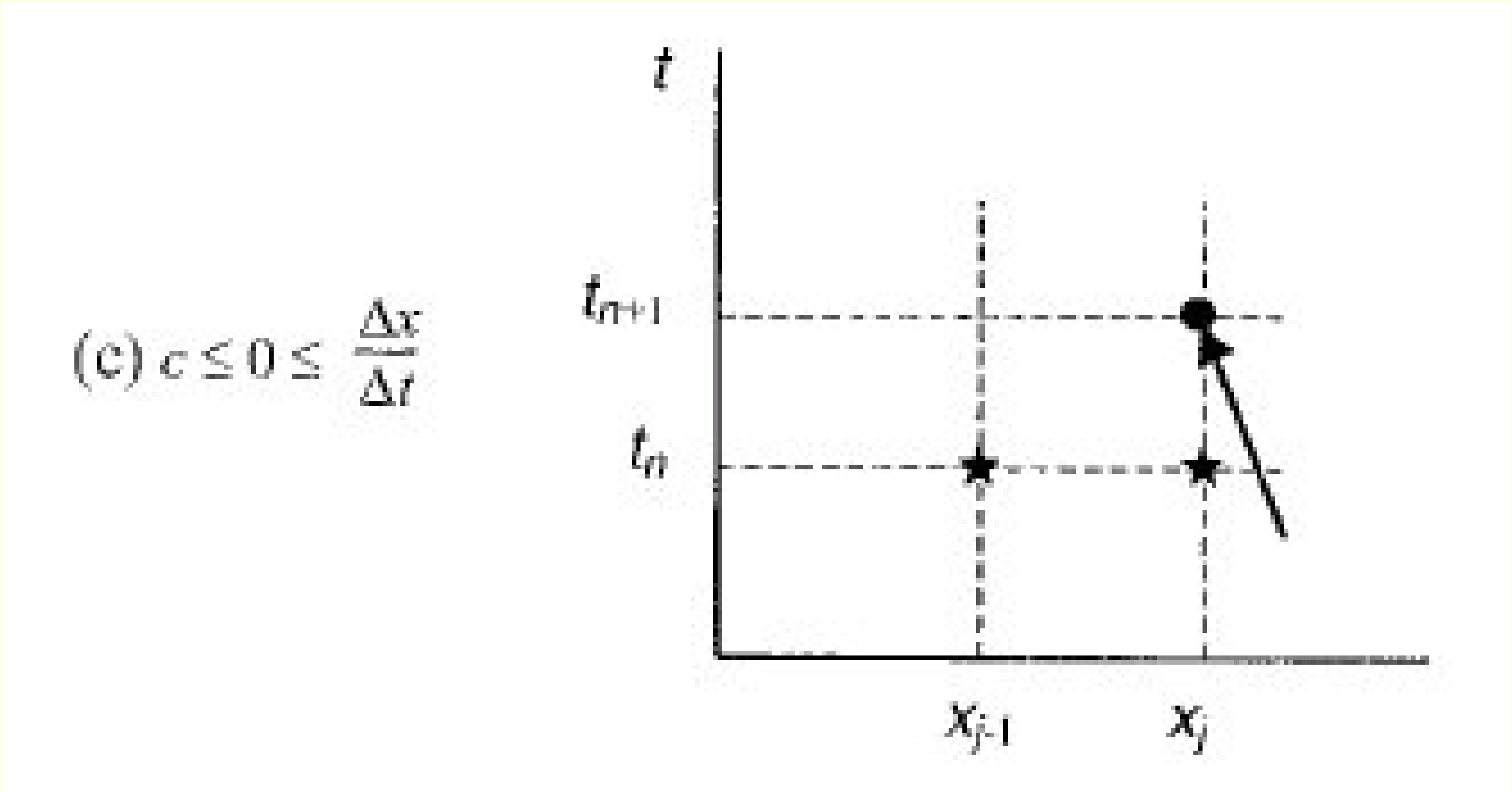

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Now defining $\Upsilon^n = \max_j |U_j^n|$, we have

$$\Upsilon^{n+1} \le \{|1-\mu| + |\mu|\}\Upsilon^n$$

and $\Upsilon^{n+1} \le \Upsilon^n$ if and only if $0 \le \mu \le 1$.

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In practice, if the condition $0 \leq \mu \leq 1$ is not satisfied, the FDE blows up in a few time steps.

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Taking absolute values of the FDE we get

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Now defining $\Upsilon^n = \max_j |U_j^n|$, we have

$$\Upsilon^{n+1} \leq \{|1-\mu| + |\mu|\}\Upsilon^n$$

and $\Upsilon^{n+1} \leq \Upsilon^n$ **if and only if** $0 \leq \mu \leq 1$.

If the condition $0 \leq \mu \leq 1$ is **not** satisfied, then **the solution is not bounded** and it grows with n.

If we let $\Delta t \to 0$ and $\Delta x \to 0$ with $\mu = const.$, it only makes things worse, because then $n \to \infty$.

In practice, if the condition $0 \leq \mu \leq 1$ is not satisfied, the FDE **blows up** in a few time steps.

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Do you believe me? See the following exercise.

Practical Exercise:

Use the simple model `SLAM` to explore the phenomenon of computational instability.

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Break here

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Recall the story of Courant, Friedrichs and Lewy in Göttingen.

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Example: **We use the criterion of the maximum method to study the stability condition of the diffusion equation**

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A FDE approximation (FTCS scheme) is given by

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \sigma \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2}$$

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We can write the FDE in the form

$$U_j^{n+1} = \mu U_{j+1}^n + (1 - 2\mu) U_j^n + \mu U_{j-1}^n$$

where $\mu = \sigma \Delta t / \Delta x^2$.

Again,

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We need a more powerful method of establishing stability.

The von Neumann Method

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The space variable x and the wavenumber k can be multi-dimensional, e.g., $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{k} = (k_1, k_2, k_3)$ but, for simplicity, we will consider the scalar case.

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Then the Fourier expansion is

$$U_j^n = \sum_p Z_p^n e^{ipj} \qquad \text{(Note: } kx = kj\Delta x = pj\text{)}$$

When we substitute this Fourier expansion into a linear FDE, we obtain a system of equations

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Therefore, stability is guaranteed if the factor ρ^n is bounded for all p when $\Delta t \to 0$ and $n \to \infty$.

So, we must have $|\rho_p|^n < M$ for all p as $n \to \infty$.

Aside: Spectral Radius

For the multi-dimensional case, the modulus of ρ is replaced by the norm of the matrix G and the stability condition becomes $||\mathbf{G}^n|| < M$ for all p, as $n \to \infty$.

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End of digression

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This is the von Neumann necessary condition for computational stability.

Comment: **The term $O(\Delta t)$ allows for bounded growth which may arise from a physical instability present in the PDE.**

If the exact solution grows with time, then the FDE cannot both satisfy $|\rho| \leq 1$ and be consistent with the PDE.

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For more complicated equations, the von Neumann criterion involves a matrix G rather than the amplification factor ρ.

The stability criterion then involves the eigenvalues of the amplification matrix, and the von Neumann stability criterion is $||\mathbf{G}|| \leq 1 + O(\Delta t)$.

Application to Advection Equation

PDE: $\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0$

FDE: $\dfrac{U_j^{n+1} - U_j^n}{\Delta t} + c\dfrac{U_j^n - U_{j-1}^n}{\Delta x} = 0$ (**upstream scheme**)

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$$U_j^n = \sum_p Z_p^n e^{ipj} = \sum_p A_p \rho_p^n e^{ipj}$$

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Since the equation is linear we can consider a single term

$$U_j^n = A_p \rho_p^n e^{ipj} = A\rho^n e^{ipj}$$

We substitute $U_j^n = A\rho^n e^{ipj}$ in the equation and divide by U_j^n to obtain

$$\frac{\rho - 1}{\Delta t} + c\,\frac{\left(1 - e^{-ip}\right)}{\Delta x} = 0 \quad \textbf{for all } p$$

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We need to estimate the maximum value of ρ.

$$\rho = 1 - \mu(1 - e^{-ip}) = 1 - \mu(1 - \cos p + i \sin p)$$

Then the modulus squared is just

$$|\rho|^2 = [1 - \mu(1 - \cos p)]^2 + \mu^2 \sin^2 p$$

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We make use of the trigonometrical relationships

$$\cos p = \cos^2 \frac{p}{2} - \sin^2 \frac{p}{2} = c^2 - s^2 \qquad \sin p = 2 \sin \frac{p}{2} \cos \frac{p}{2} = 2sc$$

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Substituting these into $|\rho|^2$ we have

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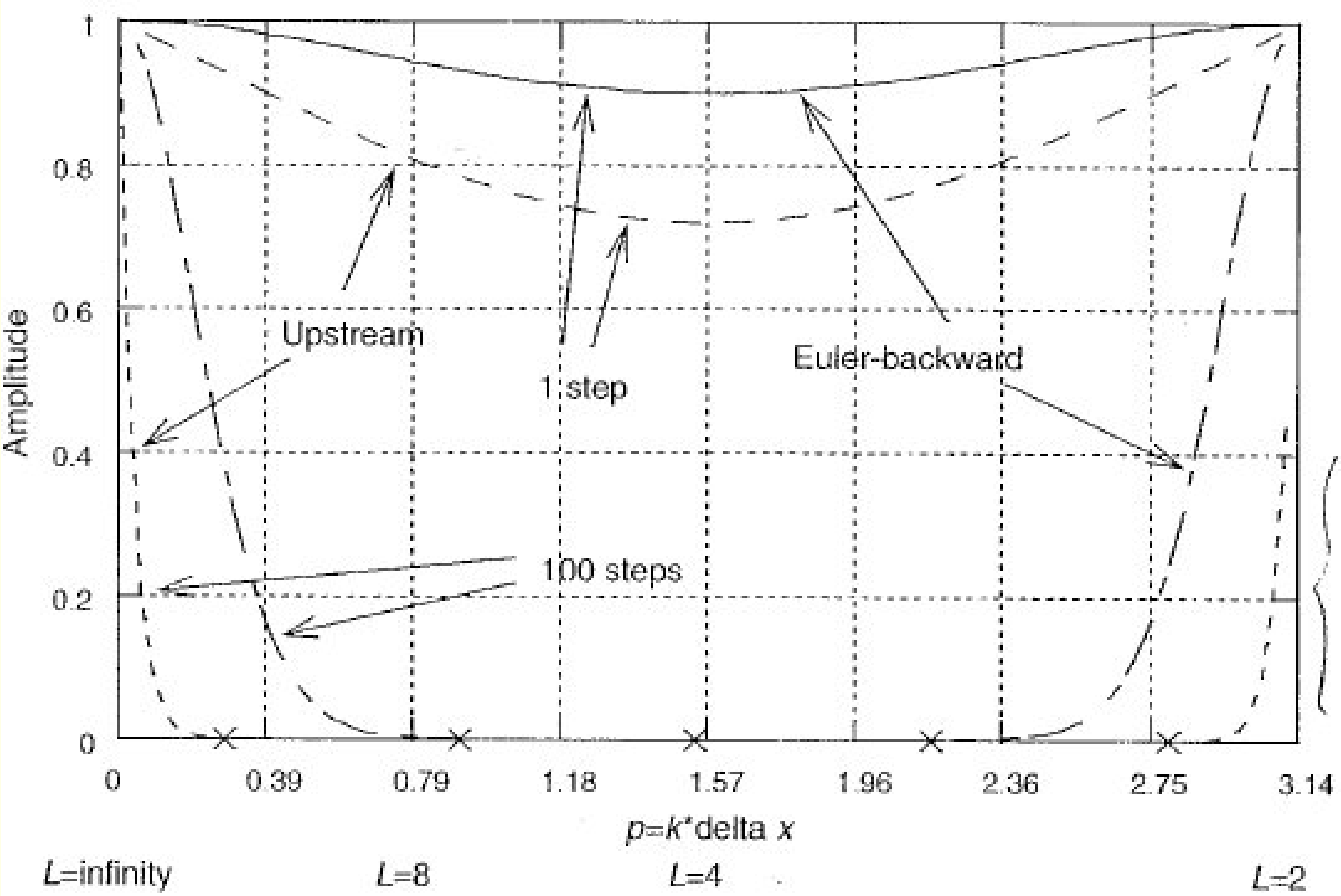
An alternative, less damping scheme known as the Matsuno or Euler-backward scheme is also shown.



Amplification factor for the upstream scheme and the Matsuno scheme, with Courant Number $\mu = 0.5$. Response for 1 step and 100 steps shown. L is the wavelength in units of Δx.