Laplace Transform Integration (ACM 40520)

Peter Lynch

School of Mathematical Sciences

Outline

- Basic Theory
- Residue Theorem
- Ordinary Differential Equations
- General Vector NWP Equation
- Lagrangian Formulation

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Basic Theory

Residue Theorem

Ordinary Differential Equations

General Vector NWP Equation

Lagrangian Formulation

The Laplace Transform: Definition

For a function of time $f(t)$, the LT is defined as

$$\hat{f}(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t\,.$$

Here, s is complex and $\hat{f}(s)$ is a complex function of s.

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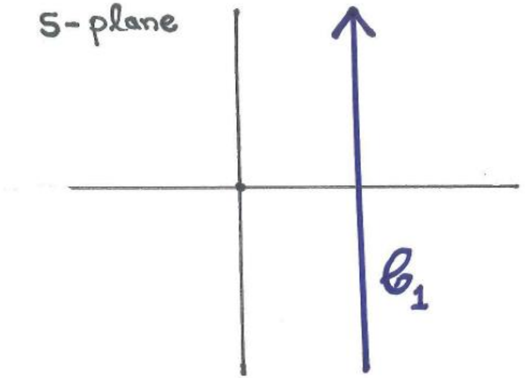
The inversion from $\hat{f}(s)$ back to $f(t)$ is

$$f(t) = \frac{1}{2\pi i}\int_{\mathcal{C}_1} e^{st}\,\hat{f}(s)\,\mathrm{d}s\,.$$

where $\mathcal{C}_1$ is a contour in the s-plane, parallel to the imaginary axis, to the right of all singularities of $\hat{f}(s)$.

Contour for inversion of Laplace Transform



Integral Transforms

The LT is one of a large family of integral transforms

They can be defined as

$$F(s) = \int_{\mathcal{R}} K(s,t)\, f(t)\, \mathrm{d}t$$

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For example, the Fourier transform is

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}\, f(t)\, \mathrm{d}t \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{+i\omega t}\, \tilde{f}(\omega)\, \mathrm{d}\omega$$

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The Hilbert transform is another . . . and many more.

The LT is a linear operator

$$\mathcal{L}\{f(t)\} = \hat{f}(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t\,.$$

Therefore

$$\mathcal{L}\{\alpha f(t)\} = \int_0^\infty e^{-st}\,\alpha f(t)\,\mathrm{d}t = \alpha \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \alpha\mathcal{L}\{f(t)\}\,.$$

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Also

$$\mathcal{L}\{f(t)+g(t)\} = \int_0^\infty e^{-st}\,[f(t)+g(t)]\,\mathrm{d}t = \mathcal{L}\{f(t)\}+\mathcal{L}\{g(t)\}\,.$$

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Therefore

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\}\,.$$

Basic Properties of the LT

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All these results can be demonstrated immediately by using the definition of the Laplace transform $\mathcal{L}\{f(t)\}$.

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A Simple Oscillation

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A pure (monochrome) oscillation in time transforms to a function with a single pole.

The position of the pole is determined by the frequency of the oscillation.

Again

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We augment $\mathcal{C}_1$ by a semi-circular arc $\mathcal{C}_2$ in the left half-plane. Denote the resulting closed contour by $\mathcal{C}_0 = \mathcal{C}_1 \cup \mathcal{C}_2$.

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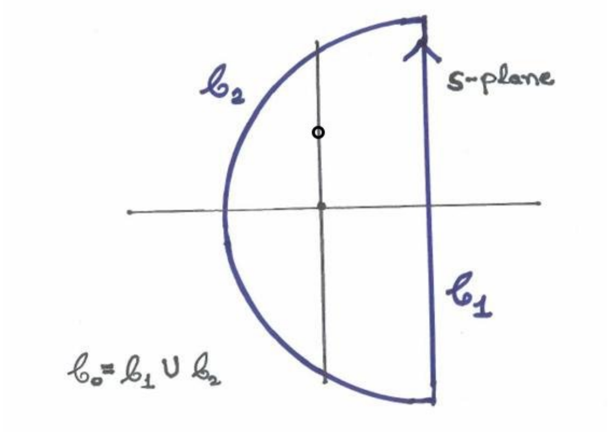
See textbook of Doetsch

Closed Contour



For an integral around a closed contour,

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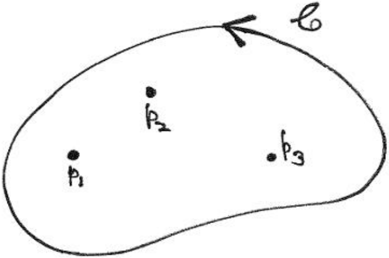
we can apply the residue theorem:

$$f(t) = \sum_{\mathcal{C}_0} \left[\text{Residues of } \left(\frac{\alpha \exp(st)}{s - i\omega} \right) \right]$$

so $f(t)$ is the sum of the residues of the integrand within $\mathcal{C}_0$.

Residue Theorem



$$\oint_{\mathcal{C}} f(z)\,dz = \left[\begin{array}{l}\text{Sum of Residues of} \\ f(z) \text{ at poles within } \mathcal{C}\end{array}\right]$$

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There is just one pole, at $s = i\omega$. The residue is

$$\lim_{s \to i\omega} (s - i\omega) \left(\frac{\alpha \exp(st)}{s - i\omega} \right) = \alpha \exp(i\omega t)$$

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So we recover the input function:

$$f(t) = \alpha \exp(i\omega t)$$

A Two-Component Oscillation

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which has two simple poles, at $s = i\omega$ and $s = i\Omega$.

The **LF pole**, at $s = i\omega$, is close to the origin.

The **HF pole**, at $s = i\Omega$, is far from the origin.

Again

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The inverse transform of $\hat{f}(s)$ is

$$\begin{aligned} f(t) &= \frac{1}{2\pi i} \oint_{\mathcal{C}_0} \frac{a \exp(st)}{s - i\omega} \mathrm{d}s + \frac{1}{2\pi i} \oint_{\mathcal{C}_0} \frac{A \exp(st)}{s - i\Omega} \mathrm{d}s \\ &= a \exp(i\omega t) + A \exp(i\Omega t) . \end{aligned}$$

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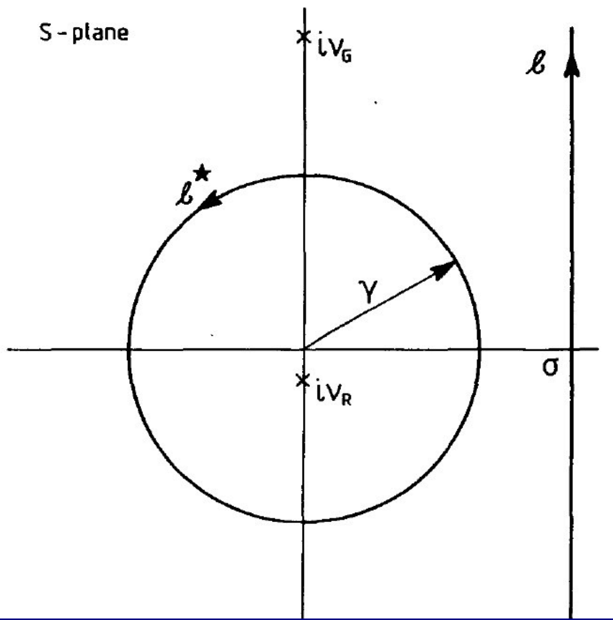
We now replace $\mathcal{C}_0$ by a circular contour $\mathcal{C}^\star$ centred at the origin, with radius γ such that $|\omega| < \gamma < |\Omega|$.


S - plane
iv_G
ℓ
ℓ^*
γ
σ
iv_R

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Therefore,

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We have filtered $f(t)$: the function $f^\star(t)$ is the LF component of $f(t)$. **The HF component is gone.**

Exercise

Consider the test function

$$f(t) = \alpha_1 \cos(\omega_1 t - \psi_1) + \alpha_2 \cos(\omega_2 t - \psi_2) \qquad |\omega_1| < |\omega_2|$$

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Show that the LT is

$$\hat{f}(s) = \frac{\alpha_1}{2}\left[\frac{e^{-i\psi_1}}{s - i\omega_1} + \frac{e^{i\psi_1}}{s + i\omega_1}\right] + \frac{\alpha_2}{2}\left[\frac{e^{-i\psi_2}}{s - i\omega_2} + \frac{e^{i\psi_2}}{s + i\omega_2}\right]$$

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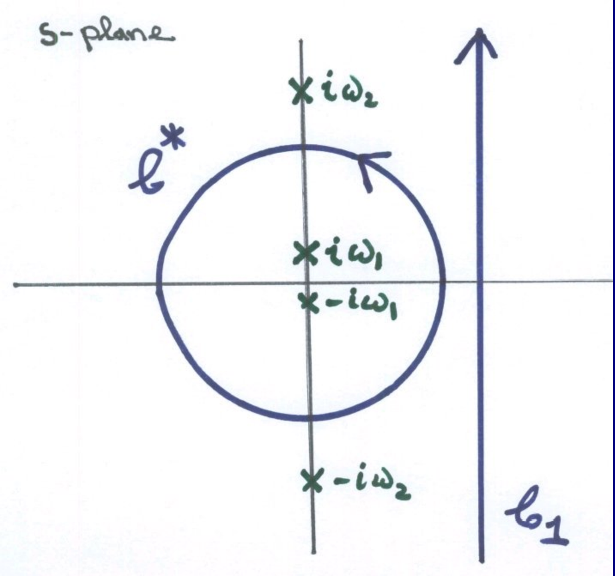
Show how, by choosing $\mathcal{C}^\star$ with $|\omega_1| < \gamma < |\omega_2|$, the HF component can be eliminated.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We can immediately solve for the transform solution:

$$\hat{w}(s) = \frac{1}{s + i\omega}\left[w_0 - \frac{n_0}{s}\right]$$

Using partial fractions, we write the transform as

$$\hat{w}(s) = \left(\frac{w_0}{s + i\omega} \right) + \frac{n_0}{i\omega} \left(\frac{1}{s} - \frac{1}{s + i\omega} \right)$$

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Thus, the solution is

$$w^\star(t) = \begin{cases} \left(w_0 - \frac{n_0}{i\omega}\right)\exp(-i\omega t) + \frac{n_0}{i\omega} & : \quad |\omega| < \gamma \\ \frac{n_0}{i\omega} & : \quad |\omega| > \gamma \end{cases}$$

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For a HF oscillation ($|\omega| > \gamma$), the solution contains only a constant term.

Thus, high frequencies are filtered out.

Outline

A General Vector Equation

We write the general NWP equations symbolically as

$$\frac{d\mathbf{X}}{dt} + i\,\mathbf{L}\mathbf{X} + \mathbf{N}(\mathbf{X}) = \mathbf{0}$$

where $\mathbf{X}(t)$ is the state vector at time t.

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We apply the Laplace transform to get

$$(s\,\hat{\mathbf{X}} - \mathbf{X}_0) + i\,\mathbf{L}\hat{\mathbf{X}} + \frac{1}{s}\mathbf{N}_0 = \mathbf{0}$$

where $\mathbf{X}_0$ is the initial value of $\mathbf{X}$ and $\mathbf{N}_0 = \mathbf{N}(\mathbf{X}_0)$ is held constant at its initial value.

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The frequencies are entangled. How do we proceed?

Eigenanalysis

$$\dot{\mathbf{X}} + i\,\mathbf{L}\mathbf{X} + \mathbf{N}(\mathbf{X}) = \mathbf{0}$$

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$$\mathbf{L}\mathbf{E} = \mathbf{E}\Lambda$$

where $\Lambda = \mathbf{diag}(\lambda_1, \ldots, \lambda_N)$ **and** $\mathbf{E} = (\mathbf{e}_1, \ldots, \mathbf{e}_N)$.

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More explicitly, assume that the eigenfrequencies split in two:

$$\Lambda = \begin{bmatrix} \Lambda_Y & \mathbf{0} \\ \mathbf{0} & \Lambda_Z \end{bmatrix}$$

Λ_Y **: Frequencies of rotational modes (LF)**

Λ_Z **: Frequencies of gravity-inertia modes (HF)**

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This equation separates into two sub-systems:

$$\begin{aligned} \dot{\mathbf{Y}} + i\,\Lambda_Y\mathbf{Y} + \mathbf{N}_Y(\mathbf{Y},\mathbf{Z}) &= 0 \\ \dot{\mathbf{Z}} + i\,\Lambda_Z\mathbf{Z} + \mathbf{N}_Z(\mathbf{Y},\mathbf{Z}) &= 0 \end{aligned}$$

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The variables Y and Z are all coupled through the nonlinear terms $\mathbf{N}_Y(\mathbf{Y},\mathbf{Z})$ and $\mathbf{N}_Z(\mathbf{Y},\mathbf{Z})$.

We first consider a single component w:

$$\dot{w} + i\lambda w + n(\mathbf{W}) = 0$$

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Holding the nonlinear term constant, we get

$$(s\hat{w} - w_0) + i\lambda\hat{w} + \frac{n_0}{s} = 0$$

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This has *two poles*, at $s = 0$ and $s = -i\lambda$:

$$\hat{w}(s) = \frac{w_0 + n_0/i\lambda}{s + i\lambda} - \frac{n_0/i\lambda}{s}$$

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If $|\lambda|$ is small, both poles are within $\mathcal{C}^\star$, so

$$w^\star(t) = \mathcal{L}^\star\{\hat{w}\} = \left(w_0 + \frac{n_0}{i\lambda}\right) e^{-i\lambda t} - \left(\frac{n_0}{i\lambda}\right) ,$$

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This corresponds to putting $\dot{w} = 0$ in the equation:

$$i\lambda w^\star + n_0 = 0$$

General Solution Method

We recall that the Laplace transform of the equation is

$$(s\hat{\mathbf{X}} - \mathbf{X}_0) + i\mathbf{L}\hat{\mathbf{X}} + \frac{1}{s}\mathbf{N}_0 = \mathbf{0}$$

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Further details are given in Clancy and Lynch, 2011a,b

Outline

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Residue Theorem

Ordinary Differential Equations

General Vector NWP Equation

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where advection is now included in the time derivative.

We *re-define* the Laplace transform to be the integral in time *along the trajectory of a fluid parcel*:

$$\hat{\mathbf{X}}(s) \equiv \int_{\mathcal{T}} e^{-st}\,\mathbf{X}(t)\,\mathrm{d}t$$

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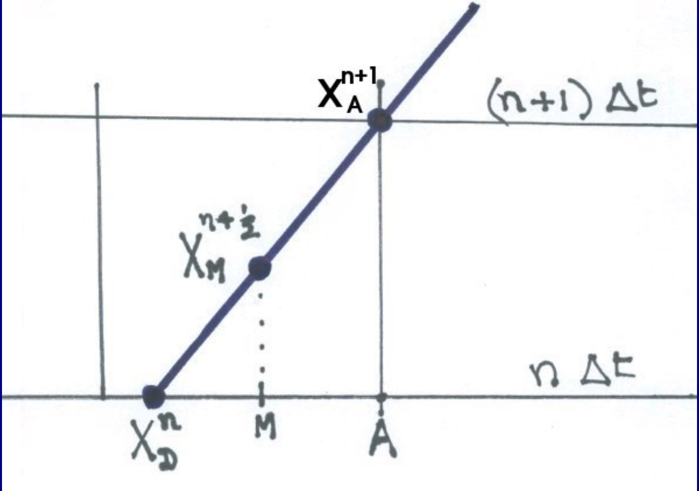
The equations thus transform to

$$(s\,\hat{\mathbf{X}} - \mathbf{X}_{\mathrm{D}}^{n}) + i\,\mathbf{L}\hat{\mathbf{X}} + \frac{1}{s}\mathbf{N}_{\mathrm{M}}^{n+\frac{1}{2}} = \mathbf{0}$$

where we evaluate nonlinear terms at a mid-point, interpolated in space and extrapolated in time.



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