The ENIAC Integrations

Numerical Solution of the BVE

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Outline

Contents

The dynamical behaviour of planetary waves in the atmosphere is modelled by the barotropic vorticity equation (BVE):

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Rossby (1939) used a simplified (linear) form of this equation for his study of atmospheric waves.

Charney, Fjørtoft & von Neumann (1950) integrated the BVE to produce the earliest numerical weather predictions on the ENIAC.

They integrated the equation on a rectangular domain, in planar geometry.

Contents

$$\mathbf{V} = \mathbf{k} \times \nabla\psi \qquad \nabla \cdot \mathbf{V} = 0$$

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$$\begin{aligned}
\frac{\mathrm{d}\bullet}{\mathrm{d}t} &= \frac{\partial\bullet}{\partial t} + u\frac{\partial\bullet}{\partial x} + v\frac{\partial\bullet}{\partial y} \\
&= \frac{\partial\bullet}{\partial t} - \frac{\partial\psi}{\partial y}\frac{\partial\bullet}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\bullet}{\partial y} \\
&= \frac{\partial\bullet}{\partial t} + J(\psi, \bullet)
\end{aligned}$$

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$$\nabla \cdot \mathbf{V} = \mathbf{0} \qquad \zeta = \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}$$

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Once initial and boundary values are given, the equation can be solved for $\psi = \psi(x, y, t)$.

The Jacobian operator is defined as

$$J(\psi,\zeta) = \left(\frac{\partial\,\psi}{\partial x}\frac{\partial\zeta}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\,\zeta}{\partial x} \right)$$

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The Jacobian operator represents advection:

$$\begin{aligned}
\mathbf{V}\cdot\nabla\zeta &= u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial y} \\
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It is essentially nonlinear. The BVE must be solved by numerical means. We come to this next.

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We write the system of equations

$$\zeta = \nabla^2\psi \qquad (1)$$

$$\frac{\partial\zeta}{\partial t} = -J(\psi, \zeta + f) \qquad (2)$$

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We assume that the values of $\psi(x, y)$ **on the boundary remain unchanged during the integration.**

ALGORITHM:

-
- **Given:** $\psi^n(x, y)$ **at time** $t = n\Delta t$**.**
-
- **Compute** $\zeta^n(x, y)$ **using (1).**
-
- **Solve (2) for** $(\partial\zeta/\partial t)^n$**.**
-
- **Solve (3) with homogeneous boundary conditions for** $(\partial\psi/\partial t)^n$**.**
-
- **Advance** ψ **to time** $t = (n + 1)\Delta t$ **using**
  $$\psi^{n+1} = \psi^{n-1} + 2\Delta t(\partial\psi/\partial t)^n.$$

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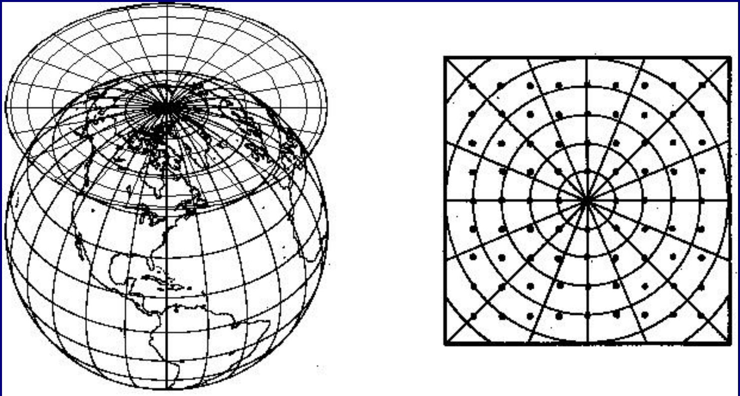

Figure: Polar Stereographic projection

Map Factor $$\mu = \frac{1}{1+\sin\phi}$$

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We introduce a discrete grid

$$\begin{aligned} x &\longrightarrow \{x_0, x_1, x_2, \ldots, x_M = M\Delta x\} \\ y &\longrightarrow \{y_0, y_1, y_2, \ldots, y_N = N\Delta y\} \end{aligned}$$

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For simplicity, we assume

$$\Delta x = \Delta y = \Delta s \, .$$

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For simplicity, we assume

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We use a spectral method that was devised by John von Neumann for the ENIAC integrations.

We recall some properties of the Fourier expansion:

$$\Phi_{mn} = \sum_{k=1}^{M-1} \sum_{\ell=1}^{N-1} \tilde{\Phi}_{k\ell} \sin\left(\frac{km\pi}{M}\right) \sin\left(\frac{\ell n\pi}{N}\right)$$

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The inverse transform is

$$\tilde{\Phi}_{k\ell} = \left(\frac{2}{M}\right)\left(\frac{2}{N}\right) \sum_{i=1}^{M-1} \sum_{j=1}^{N-1} \Phi_{ij} \sin\left(\frac{ik\pi}{M}\right) \sin\left(\frac{j\ell\pi}{N}\right)$$

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We note that

$$\sum_{i=1}^{M-1} \sum_{j=1}^{N-1} \sin\left(\frac{im\pi}{M}\right) \sin\left(\frac{jn\pi}{N}\right) \sin\left(\frac{km\pi}{M}\right) \sin\left(\frac{\ell n\pi}{N}\right)$$

$$= \delta_{ik}\delta_{j\ell}\left(\frac{M}{2}\right)\left(\frac{N}{2}\right)$$

The usual five-point approximation to $\nabla^2\Phi$ is

$$(\nabla^2\Phi)_{mn} \approx \left(\frac{\Phi_{m+1,n} + \Phi_{m-1,n} + \Phi_{m,n+1} + \Phi_{m,n-1} - 4\Phi_{m,n}}{\Delta s^2}\right)$$

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We expand Φ in a double Fourier series

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We use approximations like the following:

$$\frac{\partial^2}{\partial x^2}\sin\left(\frac{km\pi}{M}\right) \approx -4\sin^2\left(\frac{k\pi}{2M}\right)\sin\left(\frac{km\pi}{M}\right)$$

[Exercise: confirm the details.]

Thus:

$$\nabla^2 \sin\left(\frac{km\pi}{M}\right)\sin\left(\frac{\ell n\pi}{N}\right) \approx$$
$$-\frac{4}{\Delta s^2}\left[\sin^2\left(\frac{k\pi}{2M}\right)+\sin^2\left(\frac{\ell\pi}{2N}\right)\right]\sin\left(\frac{km\pi}{M}\right)\sin\left(\frac{\ell n\pi}{N}\right)$$

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The Laplacian is applied term-by-term to Φ:

$$\nabla^2\Phi_{mn} \approx$$
$$-\frac{4}{\Delta s^2}\sum_{k=1}^{M-1}\sum_{\ell=1}^{N-1}\left[\sin^2\left(\frac{k\pi}{2M}\right)+\sin^2\left(\frac{\ell\pi}{2N}\right)\right]\tilde{\Phi}_{k\ell}\times$$
$$\sin\left(\frac{km\pi}{M}\right)\sin\left(\frac{\ell n\pi}{N}\right)$$

We now expand the right hand side function:

$$F_{mn} = \sum_{k=1}^{M-1} \sum_{\ell=1}^{N-1} \tilde{F}_{k\ell} \sin\left(\frac{km\pi}{M}\right) \sin\left(\frac{\ell n\pi}{N}\right)$$

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Now we equate the coefficients of $\nabla^2\Phi$ and F:

$$\left[\sin^2\left(\frac{k\pi}{2M}\right) + \sin^2\left(\frac{\ell\pi}{2N}\right)\right] \tilde{\Phi}_{k\ell} = (-\Delta s^2/4)\tilde{F}_{k\ell}$$

or

$$\tilde{\Phi}_{k\ell} = \frac{(-\Delta s^2/4)\tilde{F}_{k\ell}}{\sin^2\left(\frac{k\pi}{2M}\right) + \sin^2\left(\frac{\ell\pi}{2N}\right)}$$

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Now $\tilde{\Phi}_{k\ell}$ is known, and we can invert it:

$$\Phi_{mn} = \frac{\Delta s^2}{MN} \sum_{k=1}^{M-1} \sum_{\ell=1}^{N-1} \tilde{\Phi}_{k\ell} \sin\left(\frac{km\pi}{M}\right) \sin\left(\frac{\ell n\pi}{N}\right)$$

We can compute the inverse transform in one go:

$$\Phi_{mn} = -\frac{\Delta s^2}{MN}\sum_{i=1}^{M-1}\sum_{j=1}^{N-1}\sum_{k=1}^{M-1}\sum_{\ell=1}^{N-1}\left[\sin^2\left(\frac{k\pi}{2M}\right)+\sin^2\left(\frac{\ell\pi}{2N}\right)\right]^{-1}\times$$
$$F_{ij}\sin\left(\frac{im\pi}{M}\right)\sin\left(\frac{jn\pi}{N}\right)\sin\left(\frac{km\pi}{M}\right)\sin\left(\frac{\ell n\pi}{N}\right)$$

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We now substitute

$$F_{ij} \longrightarrow \left(\frac{\partial\zeta}{\partial t}\right)_{ij}.$$

Then

$$\Phi_{mn} = \left(\frac{\partial\psi}{\partial t}\right)_{mn}$$

and we have the solution for Φ.

The equation

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Charney, Fjørtoft and von Neumann (*Tellus*, 1950) used z rather than ψ. This necessitates an approximation involving the β-term.

Lynch (*BAMS*, 2008) showed that the ψ-form yields forecasts that are slightly more accurate.

This confirmed a hypothesis advanced earlier by Norman Phillips.

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The ENIAC forecasts had an "electrifying effect" on the meteorological community, and led ultimately to operational NWP.

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