M.Sc. in Computational Science

Fundamentals of Atmospheric Modelling

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Lecture 7

Potential Vorticity Conservation

Applications of PV Conservation

We consider a few simple applications of Potential Vorticity conservation. The treatment is purely qualitative.

A quantitative treatment will be undertaken in subsequent lectures.

- ***Gravity-Inertia Waves.***
- ***Free Rossby Waves.***
- ***Forced Rossby Waves. Lee-side Trough.***

The Potential Vorticity Equation

Recall that the continuity equation may be written:

$$\frac{dh}{dt} + h\delta = 0$$

The vorticity equation may be written:

$$\frac{d}{dt}(\zeta + f) + (\zeta + f)\delta = 0\,.$$

Taking logarithms, we may write these in the form

$$\frac{d}{dt}\log(\zeta + f) = -\delta\,, \qquad \frac{d}{dt}\log h = -\delta\,.$$

We eliminate δ between these equations to get:

$$\frac{d}{dt}\log(\zeta + f) - \frac{d}{dt}\log h = 0\,.$$

This may also be put in the following form:

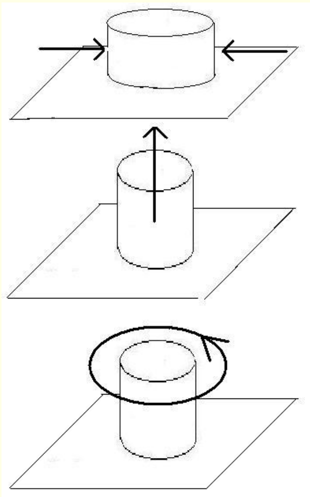
$$\frac{d}{dt}\left(\frac{\zeta + f}{h}\right) = 0\,.$$

This is the equation of *conservation of potential vorticity*.

Elementary Applications of PV Conservation

Gravity Waves: A First Look

Suppose initially the flow is irrotational and is converging towards a point. Assume that f is constant. Consider a column of fluid.



- *Convergence* induces stretching
- Stretching implies increased pressure at the centre
- Increasing h also implies increasing ζ
- $\zeta > 0$ implies Cyclonic flow
- Cyclonic flow around high pressure is ***unbalanced***
- PGF and Coriolis force act outwards
- $\therefore$ ***Divergent*** flow is induced.

The restoring forces give rise to ***Inertia-gravity waves***.

Rossby Waves: A First Look

Suppose initially the flow is nondivergent, so there is no vertical velocity and h is constant for a fluid parcel. Then absolute vorticity is conserved.

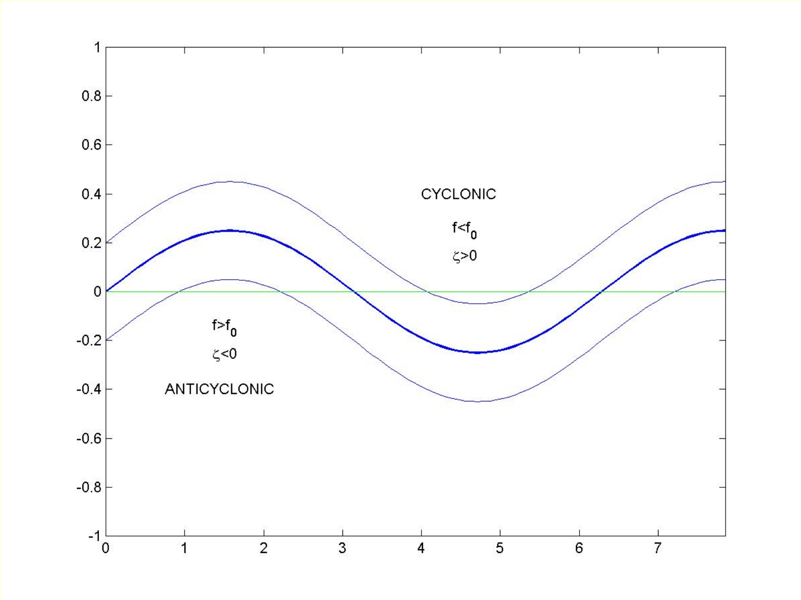
$$\frac{d(\zeta + f)}{dt} = 0\,.$$

Suppose a parcel, initially at latitude $y - y_0$ with $f = f_0$ and $\zeta = 0$. Suppose that it moves North-eastward.

- Increasing f means decreasing ζ
- Negative ζ corresponds to anticyclonic flow
- Flow curves back towards $y = y_0$
- Then $f = f_0$ and $\zeta = 0$ again
- Parcel continues SE and opposite half-cycle occurs.

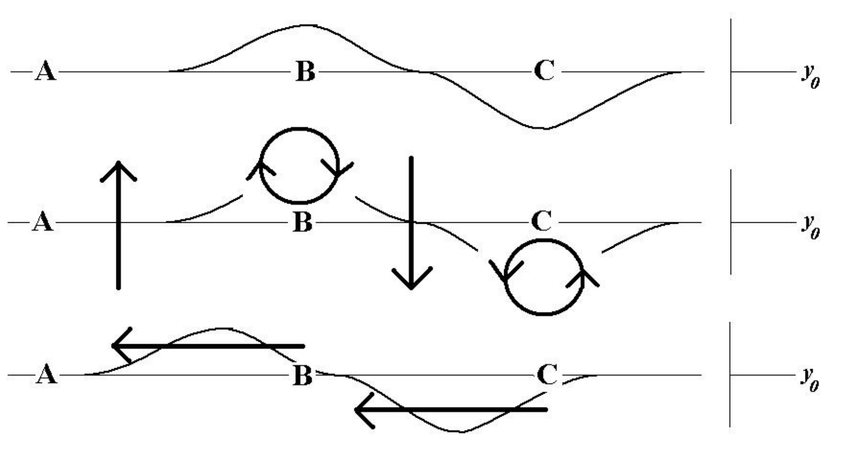
Throughout the motion, $\zeta + f$ keeps the same value.

Constant Absolute Vorticity (CAV) Trajectories.



Fluid parcels follow trajectories on which $\zeta + f$ remains constant.

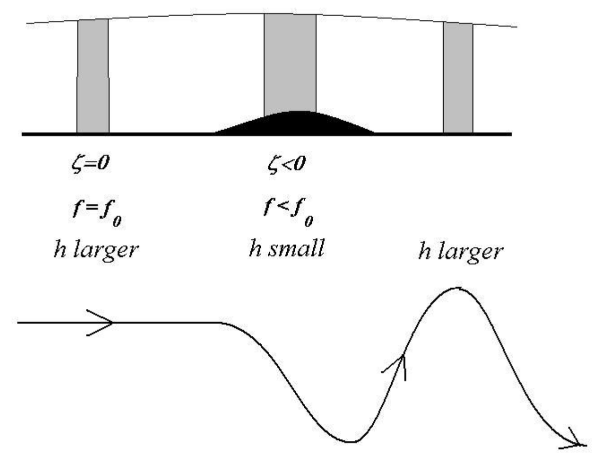
Naïve argument concerning movement of Rossby waves



Handwaving explanation of Rossby Wave Propagation.

This qualitative argument indicates *westward propagation.*

Formation of a Lee-side Trough



Formation of lee-side trough.

A mountain chain may produce a train of forced Rossby waves.

More Conservation Properties

The Circulation Theorem

The momentum equation in vector form is:

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}.\nabla\mathbf{V} + f\mathbf{k}\times\mathbf{V} + \nabla\Phi = 0 \tag{7}$$

We easily prove the following vector identity:

$$\mathbf{V}\cdot\nabla\mathbf{V} = \nabla(\tfrac{1}{2}\mathbf{V}\cdot\mathbf{V}) + \zeta\mathbf{k}\times\mathbf{V}$$

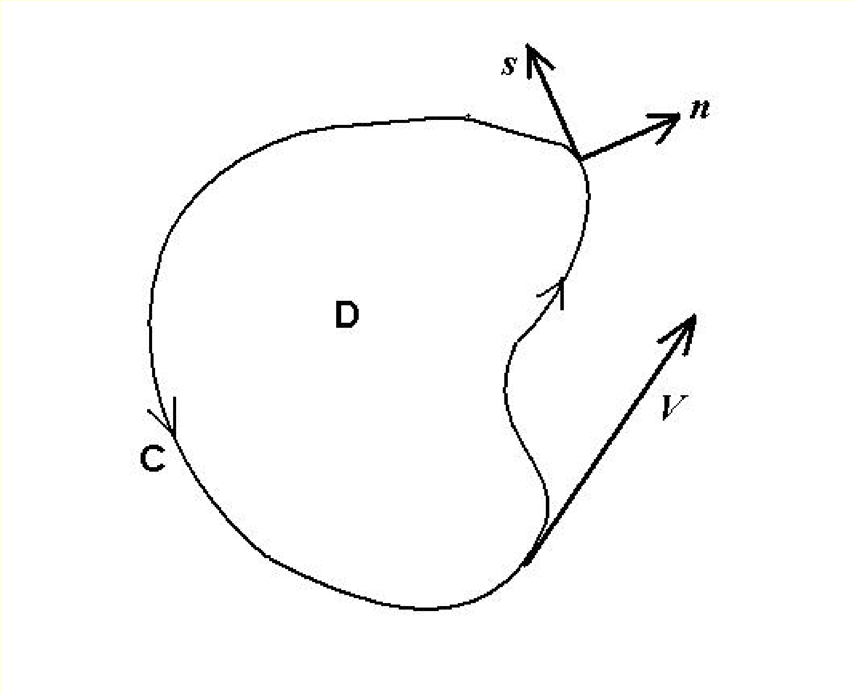
Using this, the momentum equation may be written:

$$\frac{\partial \mathbf{V}}{\partial t} + \nabla(\tfrac{1}{2}\mathbf{V}\cdot\mathbf{V}) + (f+\zeta)\mathbf{k}\times\mathbf{V} + \nabla\Phi = 0 \tag{8}$$

Assume fluid system is contained in region D with boundary C, with no flow across C. We integrate equation (8) around the contour C.

The cross-product term vanishes, because $\mathbf{k}\times\mathbf{V}$ is perpendicular to $\mathbf{V}$ and thus to $\mathbf{s}$.

The gradient terms integrate to zero, because the contour is closed.



Contour defined by the flow velocity V

Thus, the integral of (8) gives

$$\oint_C \frac{\partial \mathbf{V}}{\partial t} \cdot d\mathbf{s} = \frac{d}{dt}\oint_C \mathbf{V} \cdot d\mathbf{s} = 0$$

The integral of V around C is called the *circulation*. This result shows that the circulation around the boundary of the domain remains constant.

Alternative view: **the vorticity equation can be written**

$$\frac{\partial \zeta}{\partial t} + \nabla\cdot(\zeta + f)\mathbf{V} = 0$$

Integrating this over D:

$$\frac{d}{dt}\iint_D \zeta\, da = -\iint_D \nabla\cdot(\zeta + f)\mathbf{V}\, da = \oint_C (\zeta + f)\mathbf{V}\cdot\mathbf{n}\, ds = 0$$

Thus, the integral of vorticity over the domain is a constant. Put another way, the *average vorticity* is conserved.

Conservation of Mass.

We write the continuity equation in flux form:

$$\frac{\partial h}{\partial t} + \nabla\cdot h\mathbf{V} = 0 \qquad (9)$$

We multiply by ρ and integrate over the domain

$$\iint_D \left(\frac{\partial \rho h}{\partial t} + \nabla\cdot\rho h\mathbf{V}\right) da = 0$$

By the divergence theorem, the second term vanishes:

$$\iint_D \nabla\cdot\rho h\mathbf{V}\, da = \oint_C \rho h\mathbf{V}\cdot\mathbf{n}\, ds = 0$$

Thus we get

$$\iint_D \frac{\partial \rho h}{\partial t}\, da = \frac{d}{dt}\iint_D \rho h\, da = 0$$

But the mass of fluid over an element of area da is $dM = \rho h\, da$. Thus, the equation expresses *conservation of total mass*.

Conservation of Energy

The *potential energy* of a column of fluid is:

$$P = \int_0^h \rho g z\, dz = \tfrac{1}{2}\rho g h^2 = \left(\frac{\rho}{2g}\right)\Phi^2 .$$

The *kinetic energy* of the column is

$$K = \int_0^h \tfrac{1}{2}\rho\mathbf{V}\cdot\mathbf{V}\, dz = \tfrac{1}{2}\rho h\mathbf{V}\cdot\mathbf{V} = \left(\frac{\rho}{2g}\right)\Phi\mathbf{V}\cdot\mathbf{V} .$$

Multiply the continuity equation (9) by Φ to get

$$\left(\frac{g}{\rho}\right)\frac{\partial P}{\partial t} = \frac{\partial}{\partial t}(\tfrac{1}{2}\Phi^2) = -\Phi\nabla\cdot\boldsymbol{\Phi}\mathbf{V} .$$

Next, multiply the momentum equation

$$\frac{\partial \mathbf{V}}{\partial t} + (\zeta + f)\mathbf{k}\times\mathbf{V} + \nabla(\Phi + \tfrac{1}{2}\mathbf{V}\cdot\mathbf{V}) = 0$$

by $\Phi\mathbf{V}$ to obtain (use $\mathbf{V}\cdot\mathbf{k}\times\mathbf{V} = 0$)

$$\Phi\mathbf{V}\cdot\frac{\partial \mathbf{V}}{\partial t} + \Phi\mathbf{V}\cdot\nabla\Phi + \Phi\mathbf{V}\cdot\nabla(\tfrac{1}{2}\mathbf{V}\cdot\mathbf{V}) = 0$$

Add the continuity equation multiplied by $\frac{1}{2}\mathbf{V}\cdot\mathbf{V}$:

$$\tfrac{1}{2}\mathbf{V}\cdot\mathbf{V}\frac{\partial\Phi}{\partial t}+\tfrac{1}{2}\mathbf{V}\cdot\mathbf{V}\nabla\cdot\Phi\mathbf{V}=0$$

to obtain the expression

$$\left(\frac{g}{\rho}\right)\frac{\partial K}{\partial t}+\nabla\cdot[(\tfrac{\mathbf{1}}{\mathbf{2}}\mathbf{V}\cdot\mathbf{V})\Phi\mathbf{V}]+\nabla\cdot\Phi^2\mathbf{V}-\Phi\nabla\cdot\Phi\mathbf{V}=0\,.$$

Finally, integrate the equations for P and K over the domain:

$$\frac{d}{dt}\iint_D\left(\frac{\rho}{2g}\right)\Phi^2\,da\;=\;-\iint_D\left(\frac{\rho}{g}\right)\Phi\nabla\cdot\Phi\mathbf{V}\,da$$

$$\frac{d}{dt}\iint_D\left(\frac{\rho}{2g}\right)\Phi\mathbf{V}\cdot\mathbf{V}\,da\;=\;+\iint_D\left(\frac{\rho}{g}\right)\Phi\nabla\cdot\Phi\mathbf{V}\,da\,.$$

Adding these gives the energy conservation equation:

$$\frac{d}{dt}\iint_D\left(\frac{\rho}{2g}\right)\left[\Phi\mathbf{V}\cdot\mathbf{V}+\Phi^2\right]da=\frac{d}{dt}[K+P]=0\,.\qquad(10)$$

This is the energy principle: the sum of the kinetic plus potential energy of the fluid system remains constant.

Simplification of the PV Equation

Conservation of potential vorticity implies that the quantity $P=(\zeta+f)/\Phi$, which we call the ***potential vorticity***, is conserved following the motion.
That is, the value of P for a particular parcel of fluid remains constant as that parcel is carried along with the flow.
The conservation of potential vorticity is of great significance.

If the flow is geostrophic, PV conservation provides a ***single equation for the dynamics***.
Let us assume geostrophic flow:

$$f\mathbf{k}\times\mathbf{V}+\nabla\boldsymbol{\Phi}=\mathbf{0}\,.$$

The vorticity may then be written in terms of Φ:

$$\zeta=\mathbf{k}\cdot\nabla\times\mathbf{V}=\nabla\cdot\mathbf{V}\times\mathbf{k}=\nabla\cdot(1/f)\nabla\Phi$$

The material time derivative takes the form

$$\frac{d}{dt}=\frac{\partial}{\partial t}+u_g\frac{\partial}{\partial x}+v_g\frac{\partial}{\partial y}=\frac{\partial}{\partial t}-\frac{1}{f}\frac{\partial\Phi}{\partial y}\frac{\partial}{\partial x}+\frac{1}{f}\frac{\partial\Phi}{\partial x}\frac{\partial}{\partial y}$$

Then the potential vorticity equation (6) becomes an equation for a single dependent variable, Φ:

$$\left(\frac{\partial}{\partial t}-\frac{1}{f}\frac{\partial\Phi}{\partial y}\frac{\partial}{\partial x}+\frac{1}{f}\frac{\partial\Phi}{\partial x}\frac{\partial}{\partial y}\right)\left[\frac{\nabla\cdot(1/f)\nabla\Phi+f}{\Phi}\right]=0\,.$$

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Although the above equation can be solved numerically, it is not convenient for analysis. We will derive a more amenable form now.
Assume the flow is ***quasi-geostrophic*** and ***quasi-nondivergent***:

$$\mathbf{V}\approx\frac{1}{f}\mathbf{k}\times\nabla\Phi\,,\qquad\mathbf{V}\approx\mathbf{k}\times\nabla\psi\,.$$

To the first order of approximation, we can move the factor $1/f$ inside the differential operator:

$$\frac{1}{f}\mathbf{k}\times\nabla\Phi\approx\mathbf{k}\times\nabla\left(\frac{\Phi}{f}\right)\,.$$

Thus, the geopotential and stream function are related:

$$\Phi\approx f\psi\,.$$

Now assume the deviation of geopotential from its mean value is small:

$$\Phi=\bar{\Phi}+\Phi'\qquad\text{with}\qquad\Phi'\ll\bar{\Phi}\,.$$

We can equate Φ' with $f\psi$. Then we have

$$\frac{1}{\Phi}=\frac{1}{\bar{\Phi}(1+\Phi'/\bar{\Phi})}\approx\frac{1}{\bar{\Phi}}\left(1-\frac{\bar{\Phi}}{\bar{\Phi}}\right)\approx\frac{1}{\bar{\Phi}}\left(1-\frac{f\psi}{\bar{\Phi}}\right)$$

Thus, the potential vorticity becomes

$$P\equiv\frac{\zeta+f}{\Phi}\approx\frac{1}{\bar{\Phi}}(\zeta+f)\left(1-\frac{f\psi}{\bar{\Phi}}\right)\approx\frac{f}{\bar{\Phi}}+\frac{\zeta}{\bar{\Phi}}-\frac{f^2\psi}{\bar{\Phi}^2}$$

We can ignore the variation of f in the term containing ψ, so

$$\bar{\Phi}P \approx \zeta + f - F\psi$$

where $F \equiv f_0^2/\bar{\Phi}$ is a constant.

Next, we use the nondivergent wind in the Lagrangian derivative:

$$\begin{aligned}\frac{d\alpha}{dt} &= \frac{\partial\alpha}{\partial t} + u\frac{\partial\alpha}{\partial x} + v\frac{\partial\alpha}{\partial y} \\ &= \frac{\partial\alpha}{\partial t} + \left\{\frac{\partial\psi}{\partial x}\frac{\partial\alpha}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\alpha}{\partial x}\right\} \\ &= \frac{\partial\alpha}{\partial t} + J(\psi,\alpha)\,.\end{aligned}$$

Using this together with the approximation for P derived above, the PV-equation may be written as

$$\frac{\partial}{\partial t}\left(\nabla^2\psi - F\psi\right) + J(\psi,\nabla^2\psi) + \beta\frac{\partial\psi}{\partial x} = 0\,.$$

The Barotropic QGPV Equation

The ***barotropic, quasi-geostrophic potential vorticity equation*** (the QGPV Equation) is

$$\frac{\partial}{\partial t}\left(\nabla^2\psi - F\psi\right) + J(\psi,\nabla^2\psi) + \beta\frac{\partial\psi}{\partial x} = 0\,.$$

This is a ***single equation*** for a ***single variable***, the stream function ψ.

The simplifying assumptions have the effect of eliminating high-frequency gravity wave solutions, so that only the slow Rossby wave solutions remain.

We will study the ***wave-like solutions*** of this equation in the next lecture.

We will also study numerical solutions of this equation using a program written in MATLAB and in C.