A REMARK ON A CONJECTURE OF BORWEIN AND CHOI

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Abstract. We prove the remaining case of a conjecture of Borwein and Choi concerning an estimate on the square of the number of solutions to $n = x^2 + Ny^2$ for a squarefree integer N.

1. Introduction

We consider the positive definite quadratic form $Q(x, y) = x^2 + Ny^2$ for a squarefree integer N. Let $r_{2,N}(n)$ denote the number of solutions to $n = Q(x, y)$ (counting signs and order). In this note, we estimate

$$\sum_{n \leq x} r_{2,N}(n)^2.$$

A positive squarefree integer N is called solvable (or more classically "numerus idoneus") if $x^2 + Ny^2$ has one form per genus. Note that this means the class number of the form class group of discriminant $-4N$ equals the number of genera, 2^t, where t is the number of distinct prime factors of N. Concerning $r_{2,N}(n)$, Borwein and Choi [1] proved the following:

Theorem 1.1. *Let N be a solvable squarefree integer. Let $x > 1$ and $\epsilon > 0$. We have*

$$\sum_{n \leq x} r_{2,N}(n)^2 = \frac{3}{N}\Big(\prod_{p|2N} \frac{2p}{p+1}\Big)(x \log x + \alpha(N)x) + O(N^{\frac{1}{4}+\epsilon} x^{\frac{3}{4}+\epsilon})$$

where the product is over all primes dividing $2N$ and

$$\alpha(N) = -1 + 2\gamma + \sum_{p|2N} \frac{\log p}{p+1} + \frac{2L'(1, \chi_{-4N})}{L(1, \chi_{-4N})} - \frac{12}{\pi^2}\zeta'(2)$$

where γ is the Euler-Mascheroni constant and $L(1, \chi_{-4N})$ is the L-function corresponding to the quadratic character mod $-4N$.

Based on this result, Borwein and Choi posed the following:

Conjecture 1.2. For any squarefree N,

$$\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N}\Big(\prod_{p|2N} \frac{2p}{p+1}\Big) x \log x$$

The main result in [10] was the following.

Theorem 1.3. *Let $Q(x, y) = x^2 + Ny^2$ for a squarefree integer N with $-N \not\equiv 1 \bmod 4$. Let $r_{2,N}(n)$ denote the number of solutions to $n = Q(x, y)$ (counting signs and order). Then*

$$\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N}\Big(\prod_{p|2N} \frac{2p}{p+1}\Big) x \log x.$$

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In this note, we settle the conjecture in the remaining case, namely

Theorem 1.4. *For $-N \equiv 1 \bmod 4$, we have*
$$\sum_{n \leq x} r_{2,N}(n)^2 \sim \frac{3}{N}\Big(\prod_{p \mid 2N} \frac{2p}{p+1}\Big) x \log x.$$

2. Preliminaries

Let $Q(x,y) = ax^2 + bxy + cy^2$ denote a positive definite integral quadratic form with discriminant $D = b^2 - 4ac$ and $gcd(a,b,c) = 1$. Given Q, let κ be the largest positive integer with D/κ^2 an integer congruent to 0 or 1 modulo 4. We call κ the *conductor* of Q and set $d = D/\kappa^2$. Let $r(Q,n)$ be the number of representations of the integer n by the form Q. We now relate $r(Q,n)$ to counting the number of integral ideals of norm n in a given class in a generalized ideal class group.

Given $D = \kappa^2 d$ we consider ideals in $\mathcal{O}_K$ where $K = \mathbb{Q}(\sqrt{d})$. Let I_κ be the group of fractional ideals of $\mathcal{O}_K$ which are quotients of ideals coprime to κ and P_κ be the subgroup of fractional ideals which are quotients of principal ideals $\langle \alpha \rangle \in I_\kappa$ where $\alpha \in \mathbb{Z} + \kappa\mathcal{O}$. Then set $CL_\kappa(K) = I_\kappa / P_\kappa$. The elements of $CL_\kappa(K)$ correspond bijectively to proper equivalence classes of positive definite quadratic forms of discriminant $D = \kappa^2 d$. If the proper equivalence class of Q corresponds to the ideal class $\mathfrak{c}$, then by [3], page 219, we have
$$r(Q,n) = \sum_{r \mid \kappa} w((\kappa/r)^2 d) J(\mathfrak{c}_r, n/r^2)$$
where
$$w(D) = \begin{cases} 6 & \text{if } D = -3 \\ 4 & \text{if } D = -4 \\ 2 & \text{otherwise.} \end{cases}$$

Also $J(\mathfrak{c}_r, n)$ is the number of integral ideals of norm n in the class $\mathfrak{c}_r$ where $\mathfrak{c}_r$ is the image of $\mathfrak{c}$ under the natural homomorphism $CL_\kappa(K) \to CL_{\kappa/r}(K)$. For the form $Q(x,y) = x^2 + Ny^2$ where $-N \equiv 1 \bmod 4$, the conductor $\kappa = 2$ and so we have
$$\begin{aligned} r_{2,N}(n) &= w(-4N) J(\mathfrak{c}, n) + w(-N) J(\mathfrak{c}_2, n/4) \\ &= 2J(\mathfrak{c}, n) + w(-N) J(\mathfrak{c}_2, n/4) \end{aligned}$$
where $\mathfrak{c}_2$ is the image under $CL_2(K) \to CL_1(K)$, that is, $\mathfrak{c}_2$ is a class in the ideal class group of $K = \mathbb{Q}(\sqrt{-N})$.

We now discuss a classical result of Rankin [11] and Selberg [12] which estimates the size of Fourier coefficients of a modular form. Specifically, if $f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$ is a nonzero cusp form of weight k on $\Gamma_0(N)$, then
$$\sum_{n \leq x} |a(n)|^2 = \alpha \langle f, f \rangle x^k + O(x^{k - \frac{2}{5}})$$
where $\alpha > 0$ is an absolute constant and $\langle f, f \rangle$ is the Petersson scalar product. In particular, if f is a cusp form of weight 1, then $\sum_{n \leq x} |a(n)|^2 = O(x)$. One can adapt their result to say the following. Given two cusp forms of weight k on a suitable congruence subgroup of $\Gamma = SL_2(\mathbb{Z})$, say $f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$ and $g(z) = \sum_{n=1}^{\infty} b(n) e^{2\pi i n z}$, then

$$\sum_{n\leq x} a(n)\overline{b(n)}n^{1-k} = Ax + O(x^{\frac{3}{5}})$$

where A is a constant. In particular, if f and g are cusp forms of weight 1, then $\sum_{n\leq x} a(n)\overline{b(n)} = O(x)$.

We conclude this section with a relationship between genus characters of generalized ideal class groups and the poles of the Rankin-Selberg convolution of L-functions. Recall that a group homomorphism $\chi : I_2 \to S^1$ is an ideal class character if it is trivial on P_2, i.e.

$$\chi(\langle a\rangle) = 1$$

for $a \equiv 1 \bmod \langle 2\rangle$. Thus an ideal class character is a character on the generalized class group I_2/P_2. Recall also that a genus character (see Chapter 12, section 5 in [5]) is an ideal class character of order at most two.

Let us also recall the notion of the Rankin-Selberg convolution of two L-functions. For squarefree N, consider two ideal class characters χ_1, χ_2 for $CL_2(K)$, the generalized ideal class group of $K = \mathbb{Q}(\sqrt{-N})$ and their associated Hecke L-series

$$L_2(s,\chi_1) = \sum_{(\mathfrak{a},2)=1} \frac{\chi_1(\mathfrak{a})}{N(\mathfrak{a})^s}$$

$$L_2(s,\chi_2) = \sum_{(\mathfrak{a},2)=1} \frac{\chi_2(\mathfrak{a})}{N(\mathfrak{a})^s}$$

which converge absolutely in some right half-plane. We form the convolution L-series by multiplying the coefficients,

$$L_2(s,\chi_1\otimes\chi_2) = \sum_{(\mathfrak{a},2)=1} \frac{\chi_1(\mathfrak{a})\chi_2(\mathfrak{a})}{N(\mathfrak{a})^s}$$

The following result describes a relationship between genus characters χ and the orders of poles of $L_2(s,\chi\otimes\chi)$. The proof is similar to that of Proposition 2.4 in [10].

Proposition 2.1. *Let χ be an ideal class character for $CL_2(K)$, $-N \equiv 1 \bmod 4$, and $L_2(s,\chi)$ the associated Hecke L-series. Then χ is a genus character if and only if $L_2(s,\chi\otimes\chi)$ has a double pole at $s = 1$.*

Remark 2.2. By Proposition 2.1, if χ is a non-genus character, then $L_2(s,\chi\otimes\chi)$ has at most a simple pole at $s = 1$.

3. Proof of Theorem 1.4

Proof. As the proof is similar to that of Theorem 1.3 in [10], we sketch the relevant details. If $-N \equiv 1 \bmod 4$, then the discriminant of $K = \mathbb{Q}(\sqrt{-N})$ is $-N$. We also assume that t is the number of distinct prime factors of N and so the discriminant $-N$ also has t distinct prime factors. For $K = \mathbb{Q}(\sqrt{-N})$, consider the zeta function

$$\zeta_K(s,2) = \sum_{(\mathfrak{a},2)=1} \frac{1}{N(\mathfrak{a})^s}$$

where the sum is over those ideals $\mathfrak{a}$ of $\mathcal{O}_K$ prime to 2. We now split up $\zeta_K(s,2)$, according to the classes $\mathfrak{c}_i$ of the generalized ideal class group $CL_2(K)$, into the partial zeta functions (see page 161 of [7])

$$\zeta_{\mathfrak{c}_i}(s) = \sum_{\mathfrak{a}\in\mathfrak{c}_i} \frac{1}{N(\mathfrak{a})^s}$$

so that $\zeta_K(s,2) = \sum_{i=0}^{h_2-1} \zeta_{\mathfrak{c}_i}(s)$ where h_2 is the order of $CL_2(K)$.

Let $\mathfrak{c}$ be the ideal class in $CL_2(K)$ which corresponds to the proper equivalence class of $Q(x,y) = x^2 + Ny^2$. Now let χ be an ideal class character of $CL_2(K)$ and consider the Hecke L-series for χ, namely

$$L_2(s,\chi) = \sum_{(\mathfrak{a},2)=1} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s}.$$

We may now rewrite the Hecke L-series as

$$L_2(s,\chi) = \sum_{i=0}^{h_2-1} \chi(\mathfrak{c}_i)\zeta_{\mathfrak{c}_i}(s).$$

And so summing over all ideal class characters of $CL_2(K)$, we have

$$\sum_{\chi} \overline{\chi}(\mathfrak{c}) L_2(s,\chi) = \sum_{i=0}^{h_2-1} \zeta_{\mathfrak{c}_i}(s)\Big(\sum_{\chi} \overline{\chi}(\mathfrak{c})\chi(\mathfrak{c}_i)\Big).$$

The inner sum is nonzero precisely when $\mathfrak{c} = \mathfrak{c}_i$. Thus we have

$$\zeta_{\mathfrak{c}}(s) = \frac{1}{h_2}\sum_{\chi} \overline{\chi}(\mathfrak{c}) L_2(s,\chi)$$

and so

$$\zeta_{\mathfrak{c}}(s) = \frac{1}{h_2}(L_2(s,\chi_0) + \overline{\chi_1}(\mathfrak{c})L_2(s,\chi_1) + \cdots + \overline{\chi_{h_2-1}}(\mathfrak{c})L_2(s,\chi_{h_2-1})).$$

As χ_0 is the trivial character, $L_2(s,\chi_0) = \zeta_K(s,2)$. Comparing n^{th} coefficients, we have

$$J(\mathfrak{c},n) = \frac{1}{h_2}(a_n + b_1(n) + \cdots + b_{h_2-1}(n)).$$

where a_n is the number of integral ideals of $\mathcal{O}_K$ prime to 2 and of norm n and the b_i's are coefficients of weight 1 cusp forms (see [2]). Recall we also have

$$r_{2,N}(n) = 2J(\mathfrak{c},n) + w(-N)J(\mathfrak{c}_2, n/4)$$

and so

$$r_{2,N}(n) = \frac{2}{h_2}\Big(a_n + b_1(n) + \cdots + b_{h_2-1}(n)\Big) + w(-N)J(\mathfrak{c}_2, n/4).$$

Thus

$$\sum_{n\le x} r_{2,N}(n)^2 = \frac{4}{h_2^2}\Big(\sum_{n\le x} a_n^2 + \sum_{\substack{i\\ n\le x}} b_i(n)^2 + 2\sum_{\substack{i\\ n\le x}} a_n b_i(n) + \sum_{\substack{i\ne j\\ n\le x}} b_i(n)b_j(n)\Big) +$$
$$\frac{4}{h_2}\sum_{n\le x}\Big(a_n + b_1(n) + \cdots + b_{h_2-1}(n)\Big)w(-N)J(\mathfrak{c}_2, n/4) + \sum_{n\le x} w(-N)^2 J(\mathfrak{c}_2, n/4)^2.$$

Assume $-N \equiv 1 \bmod 8$. Applying the main theorem in [6] to the Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n^2}{n^s}$, we obtain

$$\sum_{n\le x} a_n^2 \sim Ax\log x$$

where $A = \frac{1}{2\pi^2}L(1,\chi_{-N})^2 \prod_{p|N} \frac{p}{p+1}$. As $-N$ has t distinct prime factors, we have 2^t genus characters for $CL(K)$ where $K = \mathbb{Q}(\sqrt{-N})$. By [7] (see Theorem 1, page 127), we have 2^t genus characters for $CL_2(K)$. We now must estimate $\sum_{\substack{i\\ n\le x}} b_i(n)^2$. Let us now assume that the first $2^t - 1$ terms arise from L-functions associated to genus characters. By Proposition 2.1 and an application of Perron's formula, we obtain

$$\sum_{n\le x} b_i(n)^2 \sim Ax\log x.$$

As this estimate holds for each i such that $1 \le i \le 2^t - 1$, the term $Ax\log x$ appears 2^t times in the estimate of $\sum_{n\le x} r_{2,N}(n)^2$. By Remark 2.2 and the Rankin-Selberg estimate, the remaining terms are all $O(x)$. Thus

$$\sum_{n\le x} r_{2,N}(n)^2 \sim \frac{4}{h_2^2}\Big(2^t\frac{1}{2\pi^2}L(1,\chi_{-N})^2\prod_{p|N}\frac{p}{p+1}\Big)x\log x.$$

By [4], we have $L(1,\chi_{-N}) = \frac{h\pi}{\sqrt{N}}$ where h is the class number of K and $h_2 = h$. Thus

$$\sum_{n\le x} r_{2,N}(n)^2 \sim \frac{3}{N}\Big(\prod_{p|2N}\frac{2p}{p+1}\Big)x\log x.$$

For $-N \equiv 5 \bmod 8$, we have $h_2 = 3h$ and again by [6],

$$\sum_{n\le x} a_n^2 \sim \Big(\frac{9}{2\pi^2}L(1,\chi_{-N})^2\prod_{p|N}\frac{p}{p+1}\Big)x\log x.$$

Thus

$$\sum_{n\le x} r_{2,N}(n)^2 \sim \frac{3}{N}\Big(\prod_{p|2N}\frac{2p}{p+1}\Big)x\log x.$$

□

Remark 3.1. We would like to mention another approach which confirms Theorems 1.3 and 1.4. Let $Q \in \mathbb{Z}^{2\times 2}$ be a non-singular symmetric matrix with even diagonal entries and $q(\mathbf{x}) = \frac{1}{2}Q[\mathbf{x}] = \frac{1}{2}\mathbf{x}^TQ\mathbf{x}$, $\mathbf{x} \in \mathbb{Z}^2$, the associated quadratic form in two variables. Let $r(Q,n)$ denote the number of representations of n by the quadratic form Q. Now consider the theta function

$$\theta_Q(z) = \sum_{\mathbf{x}\in\mathbb{Z}^2} e^{\pi i z Q[\mathbf{x}]}.$$

The Dirichlet series associated with the automorphic form θ_Q is

$$(4\pi)^{-1/2}\zeta_Q(\tfrac{1}{2}+s)$$

where

$$\zeta_Q(s)=\sum_{n=1}^{\infty}\frac{r(Q,n)}{n^s}=\sum_{\mathbf{x}\in\mathbb{Z}^2\setminus\{\mathbf{0}\}}q(\mathbf{x})^{-s}$$

for $\Re(s)>1$. A careful and involved application of the Rankin-Selberg method to the above Dirichlet series (see Theorems 2.1 and 5.1 in [8] and Theorem 5.2 in [9]) combined with a Tauberian argument yields the following (see Theorem 6.1 in [8])

$$\sum_{n\le x}r(Q,n)^2\sim A_Q x\log x$$

where

$$A_Q=12\frac{A(q)}{q}\prod_{p|q}\left(1+\frac{1}{p}\right)^{-1}.$$

Here $q=\det Q$ and $A(q)$ denotes the multiplicative function defined by

$$A(p^e)=2+(1-\tfrac{1}{p})(e-1)$$

where p is an odd prime, $e\ge 1$, and

$$A(2^e)=\begin{cases}1 & \text{if } e\le 1,\\ 2 & \text{if } e=2,\\ e-1 & \text{if } e\ge 3.\end{cases}$$

Let us now turn to our situation. Consider $q(\mathbf{x})=x^2+Ny^2=\frac{1}{2}\mathbf{x}^TQ\mathbf{x}$ where $Q=\begin{pmatrix}2 & 0\\ 0 & 2N\end{pmatrix}$, N squarefree. Thus $q=4N$. Suppose N has t distinct prime factors. Then $A(4N)=2^{t+1}$ and so

$$A_Q=\frac{3}{N}2^{t+1}\prod_{p|2N}\left(1+\frac{1}{p}\right)^{-1}=\frac{3}{N}\prod_{p|2N}\frac{2p}{p+1}.$$

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