1. Introduction

After receiving a Ph.D. in 2001 from Louisiana State University, I have established a strong record of research and publication. In total, I have written 32 papers in the areas of number theory, \(q\)-series, modular forms, combinatorics and algebraic \(K\)-theory. In Section 2, we briefly discuss (in reverse chronological order) a selection of my results. In Section 3, we examine some of my current research projects. For a complete list of publications, please see

\[\text{http://mathsci.ucd.ie/~osburn}\]

2. Selected Results

2.1. Rogers-Ramanujan type identities from knot theory. Two of the most important results in the theory of \(q\)-series are the classical Rogers-Ramanujan identities which state that

\[
\sum_{n \geq 0} q^{n^2+sn} (q)_n = \frac{1}{(q^{1+s}; q^5)_\infty(q^{4-s}; q^5)_\infty}
\]

where \(s = 0\) or \(1\). Here and throughout, we use the standard \(q\)-hypergeometric notation,

\[(a_1, a_2, \ldots, a_j)_n = (a_1, a_2, \ldots, a_j; q)_n := \prod_{k=1}^{n} (1 - a_1 q^{k-1})(1 - a_2 q^{k-1}) \cdots (1 - a_j q^{k-1}),\]

valid for \(n \in \mathbb{N} \cup \{\infty\}\). There has been recent interest in the appearance of these (and similar) identities in knot theory. A knot \(K\) is an embedding of a circle in \(\mathbb{R}^3\). A common method of representing a knot is via a knot diagram where at each crossing the over-strand is distinguished from the under-strand by creating a break in the strand going underneath. For example, the trefoil knot \(3_1\) is given by

![Trefoil Knot Image](image)

This is an example of an alternating knot which is simply a knot in which the over and under nature of the crossings alternates as one travels along the knot. A link consists of one or more knots (“components”) entangled with each other and an alternating link is a link in which the crossings alternate as one moves along each component. Very recently, Garoufalidis and Lê [27]
have associated the $q$-multisums $\Phi_K(q)$ and $\Phi_{-K}(q)$ to a given alternating link $K$ and its mirror $-K$. These $q$-multisums are very similar to those which appear in a conjecture of Nahm [72] and are thus called generalized Nahm sums. What is compelling is the fact that for alternating knots, the evaluation of the generalized Nahm sums leads to Rogers-Ramanujan type identities. Specifically, in Appendix D of [27], Garoufalidis and Lê (with Zagier) conjectured evaluations of $\Phi_K(q)$ for 21 knots and of $\Phi_{-K}(q)$ for 22 knots in terms of modular forms and false theta series and state “every such guess is a $q$-series identity whose proof is unknown to us”. For a positive integer $b$, we define

$$h_b = h_b(q) = \sum_{n \in \mathbb{Z}} \epsilon_b(n) q^{\frac{bn(n+1)}{2} - n}$$

where

$$\epsilon_b(n) = \begin{cases} (-1)^n & \text{if } b \text{ is odd}, \\ 1 & \text{if } b \text{ is even and } n \geq 0, \\ -1 & \text{if } b \text{ is even and } n < 0. \end{cases}$$

For an integers $p$, $a$ and $b$, let $K_p$ denote the $p$th twist knot obtained by $-1/p$ surgery on the Whitehead link and $T(a, b)$ the left-handed $(a, b)$ torus knot. The 43 conjectures from [27] are as follows:

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<tr>
<th>$K$</th>
<th>$\Phi_K(q)$</th>
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<tr>
<td>$K_p$, $p &gt; 0$</td>
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<td>$h_3$</td>
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<td>$K_p$, $p &lt; 0$</td>
<td>$h_{2</td>
<td>p</td>
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<td>$T(2, p)$, $p &gt; 0$</td>
<td>$h_{2p+1}$</td>
<td>1</td>
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Table 1.
Here, we have corrected the entries for $6_1$, $7_3$, $8_1$, $8_4$, $8_5$, $K_p$, $p < 0$ (and their mirrors) and $7_5$ in Appendix D of [27]. Note that a conjectural evaluation for $\Phi_{8_5}(q)$ is not currently known.

Three of these Rogers-Ramanujan type identities, namely

$$\Phi_{3_1}(q) = h_3, \quad \Phi_{4_1}(q) = h_3 \quad \text{and} \quad \Phi_{6_1}(q) = h_3^2$$

have been proven by Andrews [2]. Motivated by his work (and in conjunction with (2.1)), we prove the following main result in [41]:

**Theorem 2.1.** The identities in Table 1 are true.

### 2.2. Supercongruences.

The term supercongruence first appeared in Beukers’ work [8] and was the subject of the Ph.D. thesis of Coster [21]. It refers to the fact that congruences of a certain type are stronger than those suggested by formal group theory. Supercongruences have been observed in the context of number theory, mathematical physics and algebraic geometry. A motivating example in [8] and [21] is the Apéry numbers

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \left(\binom{n+k}{k}\right)^2$$

which not only satisfy [28]

$$A(mp) \equiv A(m) \pmod{p^3},$$

but the two-term supercongruence [21]

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}$$

for primes $p \geq 5$ and integers $m, r \geq 1$. In [62], we consider the sequences of numbers given by

$$s_7(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \left(\binom{n+k}{k}\right) \binom{2k}{n}$$

as well as

$$s_{18}(n) = \sum_{k=0}^{[n/3]} (-1)^k \binom{n}{k} \binom{2k}{n-k} \left(\binom{2n-3k-1}{n} + \binom{2n-3k}{n}\right),$$

with $s_{18}(0) = 1$. These “sporadic” sequences were recently discovered by Cooper [20] while performing a numerical search for sequences which appear as coefficients of series for $1/\pi$ and of series expansions in $t$ of modular forms where $t$ is a modular function. Here, the subscripts 7 and 18 are used in (2.5) and (2.6) as the associated modular function is of level 7 and 18, respectively. In [20], Cooper searched for parameters $(a, b, c, d)$ such that the recurrence relation

$$(n+1)^3 s(n+1) = (2n+1)(an^2 + an + b)s(n) - n(cn^2 + d)s(n-1),$$

with initial conditions $s(-1) = 0, s(0) = 1$, produces only integer values $s(n)$ for all $n \geq 0$. The tuple $(17, 5, 1, 0)$ corresponds to the Apéry numbers (2.2), while the tuples $(13, 4, -27, 3)$ and $(14, 6, -192, 12)$ correspond to the sequences $s_7(n)$ and $s_{18}(n)$. This search was motivated by Beukers’ [9] and Zagier’s [73] work on sequences $t(n)$ defined by
\[(n+1)^2 t(n+1) = (an^2 + an + b)t(n) - cn^2 t(n-1),\]

with initial conditions \(t(-1) = 0, t(0) = 1,\) such that \(t(n) \in \mathbb{Z}\) for all \(n \geq 0.\) Zagier’s search yielded six sequences that are not either terminating, polynomial, hypergeometric or Legendrian. These six sequences were called sporadic. Interestingly, Cooper conjectured the following congruences which are reminiscent of (2.3).

**Conjecture 2.2.** For any prime \(p \geq 3,\)

\[s_7(mp) \equiv s_7(m) \pmod{p^3}.\] \hspace{1cm} (2.7)

Likewise, for any prime \(p,\)

\[s_{18}(mp) \equiv s_{18}(m) \pmod{p^2}.\] \hspace{1cm} (2.8)

The purpose of our paper [62] is to exhibit that (2.7) and (2.8) are special cases of general two-term supercongruences. For integers \(A, B, C,\) let

\[S(n; A, B, C) = \sum_{k=0}^{n} \binom{n}{k}^A \binom{n+k}{k}^B \binom{2k}{n}^C.\]

Note that this family of sequences includes the Apéry numbers (2.2) as well as the sequence \(s_7(n).\) Our main results are the following supercongruences, the first of which, in particular, generalizes the supercongruence (2.4).

**Theorem 2.3.** Let \(A \geq 2\) and \(B, C \geq 0\) be integers. For any integers \(m, r \geq 1\) and primes \(p \geq 5,\) we have

\[S(mp^r; A, B, C) \equiv S(mp^{r-1}; A, B, C) \pmod{p^{3r}}.\]

**Theorem 2.4.** For any integers \(m, r \geq 1\) and any primes \(p,\) we have

\[s_{18}(mp^r) \equiv s_{18}(mp^{r-1}) \pmod{p^{2r}}.\]

Another instance of supercongruences is in the following situation. Let \(Q = Q_{\alpha, \beta}\) be the ordinary differential operator on \(L^2(\mathbb{R}) \otimes \mathbb{C}^2\) defined by

\[Q := \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \left( -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 \right) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left( x \frac{d}{dx} + \frac{1}{2} \right)\]

where \(\alpha, \beta\) are positive real numbers satisfying \(\alpha \beta > 1.\) The system defined by the operator \(Q\) is called the non-commutative harmonic oscillator [63]. The operator \(Q\) is positive, self-adjoint and unbounded with a discrete spectrum in which the multiplicities of the eigenvalues

\[0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots (\rightarrow \infty)\]

are uniformly bounded. Thus, one can define the spectral zeta function

\[\zeta_Q(s) := \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s}.\]

The series \(\zeta_Q(s)\) is absolutely convergent, defines a holomorphic function in \(s\) for \(\text{Re}(s) > 1\) and can be meromorphically continued to \(\mathbb{C}\) (for details, see [37], [38]). In [42], Kimoto and Wakayama discuss the Apéry-like numbers.
\[ \tilde{J}_2(n) := \sum_{k=0}^{n} (-1)^k \left( \frac{-1}{2} \right) \binom{n}{k} \]

which occur in a representation of the special value \( \zeta_Q(2) \). Similar to the Apéry numbers (2.2), these numbers satisfy the recurrence relation

\[
4n^2 \tilde{J}_2(n) - (8n^2 - 8n + 3) \tilde{J}_2(n - 1) + 4(n - 1)^2 \tilde{J}_2(n - 2) = 0,
\]

with \( \tilde{J}_2(0) = 1 \) and \( \tilde{J}_2(1) = \frac{3}{4} \), possess many interesting arithmetic properties such as

\[
\tilde{J}_2(mp^r) \equiv \tilde{J}_2(mp^{r-1}) \pmod{p^r}
\]

for integers \( m, r \geq 1 \) and primes \( p \geq 3 \) and have the modular parametrization

\[
\frac{\eta(2z)^2}{\eta(z)^{12} \eta(4z)^8} = \sum_{n=0}^{\infty} \tilde{J}_2(n) t^n
\]

where

\[
t = t(z) = 16 \frac{\eta(z)^8 \eta(4z)^{16}}{\eta^{24}(2z)}
\]

and \( \eta(z) \) is the Dedekind eta-function. Our interest concerns the following conjecture from [42].

**Conjecture 2.5.** (Kimoto-Wakayama) For primes \( p \geq 3 \),

\[
\sum_{k=0}^{p-1} \tilde{J}_2(k)^2 \equiv (-1)^{\frac{p-1}{2}} \pmod{p^3}.
\]

In [46], we prove two results, the second of which is equivalent to Conjecture 2.5. Recall that for a nonnegative integer \( r \) and \( \alpha_i, \beta_i \in \mathbb{C} \) with \( \beta_i \notin \{\ldots, -3, -2, -1\} \), the (generalized) hypergeometric series \( _{r+1}F_r \) is defined by

\[
_{r+1}F_r \left[ \begin{array}{c} \alpha_1 \alpha_2 \ldots \alpha_{r+1} \\ \beta_1 \ldots \beta_r \end{array} ; l \right] := \sum_{k=0}^{\infty} \frac{(\alpha_1)_{k}(\alpha_2)_{k} \ldots (\alpha_{r+1})_k}{(\beta_1)_k \ldots (\beta_r)_k} \frac{l^k}{k!},
\]

where \( (a)_0 := 1 \) and \( (a)_{k} := a(a+1) \cdots (a+k-1) \). This series converges for \( |l| < 1 \). Hypergeometric series are an important class of special functions which have been investigated by Gauss, Euler and Kummer and have numerous applications to the theory of differential equations, algebraic varieties and physics. Note that

\[
\tilde{J}_2(n) = _3F_2 \left[ \begin{array}{c} \frac{1}{2} \frac{1}{2} -n \\ 1 1 ; 1 \end{array} \right].
\]

**Theorem 2.6.** For primes \( p > 3 \),

\[
\sum_{x=0}^{p-1} _3F_2 \left[ \begin{array}{c} \frac{1-p}{2} \frac{1+p}{2} -x \\ 1 1 ; 1 \end{array} \right]^2 \equiv (-1)^{\frac{p-1}{2}} \pmod{p^3}
\]

and for primes \( p \geq 3 \)
\[ \sum_{x=0}^{p-1} \begin{array}{c} \frac{1}{2} \ \frac{1}{2} \ -x \\ 1 \ 1 \end{array ; 1} \binom{2}{2} \equiv (-1)^{\frac{p+1}{2}} \pmod{p^3} \].

2.3. **Multisum mock theta functions.** The Indian genius Srinivasa Ramanujan left the mathematical world with many fascinating and unproven assertions that would challenge the finest researchers throughout the 20th century. Indeed, work on Ramanujan’s mathematics continues today, as many of his deepest and most tantalizing writings were only unearthed in 1976 and have still yet to be fully explained. One such mysterious assertion concerns “mock theta functions”. In Ramanujan’s last letter to Hardy, dated January 12, 1920, he lists 17 mock theta functions such as

\[ F_1(q) := \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^n; q)_n}. \] (2.9)

Between the time of Ramanujan’s death in 1920 and the early part of the 21st century, approximately 35 other \(q\)-series were studied and deemed mock theta functions. Some were introduced by Watson, some were found in Ramanujan’s lost notebook and studied by Andrews, Choi, and Hickerson, and others were produced by Berndt, Chan, Gordon and McIntosh using intuition from \(q\)-series. For a summary of this classical work, see [29] or [34].

Surprisingly, much remained unknown about these series until very recently. Ramanujan’s claims about their analytic properties remained open and there was even discussion concerning the rigorous definition of such a function. Despite these issues, Ramanujan’s mock theta functions indeed possess many striking properties and have been the subject of an astonishing number of important works. Additionally, mock theta functions (and in general \(q\)-series) have many surprising connections and important implications in the study of characters of infinite dimensional Lie superalgebras and conformal field theory, quantum invariants of knots, links, and 3-dimensional manifolds, algebraic \(K\)-theory, homological mirror symmetry, quantum black holes and Moonshine phenomena.

Thanks to the 2002 Ph.D. thesis of Zwegers and work of Bringmann and Ono, we now know that each of Ramanujan’s original 17 (and the subsequent) examples of mock theta functions is the holomorphic part of a weight 1/2 harmonic weak Maass form with a weight 3/2 unary theta function as its “shadow”. Following Zagier, the holomorphic part of any weight \(k\) harmonic weak Maass form is called a mock modular form of weight \(k\). If \(k = 1/2\), then it is called a mock theta function. Further developments concerning mock theta functions would have striking applications not only in number theory and combinatorics, but also in the diverse areas mentioned above. For more on these functions, their remarkable history and modern developments, see [58] and [74].

A natural question is whether or not there exist other examples of \(q\)-hypergeometric series which are mock theta functions (in the modern sense of Zagier). Several authors have recently addressed this question, constructing two-variable \(q\)-series which are essentially “mock Jacobi forms” and which then specialize at torsion points to mock theta functions.

In [50], we use \(q\)-series techniques such as Bailey pairs, the Bailey chain and a “change of base” result due to Bressoud, Ismail and Stanton in combination with recent important work of Hickerson and Mortenson on Hecke-type double sums [34] to explicitly construct four infinite
families of $q$-hypergeometric multisums which are mock theta functions. For example, if we define
\[
\mathcal{R}^{(k)}(q) := \sum_{n_k \geq n_{k-1} \geq \ldots \geq n_1 \geq 0} \frac{q^{n_k^2+n_k}}{(-q)_{n_k}} \mathcal{B}(n_k,\ldots,n_1;q)
\]
where
\[
\mathcal{B}(n_k,\ldots,n_1;q) := q^{(n_k-1)+n_{k-2}+2n_{k-3}+\cdots+2k-3}(1-n_1)
\times \frac{(-q)_{n_k-1}(-q)_{2n_{k-2}}(-q^2; q^2)^{2n_{k-3}} \cdots (-q^{2k-3}; q^{2k-3})_{2n_1}}{(q)_{n_k-n_{k-1}}(q^2; q^2)_{n_{k-1}-n_{k-2}} \cdots (q^{2k-2}; q^{2k-2})_{n_2-n_1}(q^{2k-1}; q^{2k-1})_{n_1}},
\]
then one of the main results in [50] is the following.

**Theorem 2.7.** For $k \geq 3$, $\mathcal{R}^{(k)}(q)$ is a mock theta function.

As a consequence, we prove an identity between the multisum $\mathcal{R}^{(k)}(q)$ and the classical mock theta function (2.9) of “order 7”.

**Corollary 2.8.** We have $\mathcal{R}^{(3)}(q) = q^{-1} \mathcal{F}_1(q^4) + M(q)$ where $M(q)$ is an explicitly computable weakly holomorphic modular form.

Thus, one can think of Ramanujan’s mock theta functions as specializations of infinite families of mock theta functions. This is only the beginning. For example, we will study other change of bases, asymptotics and congruences for the coefficients of $\mathcal{R}^{(k)}(q)$ and the combinatorial significance (for example, lattice paths) of mock $q$-hypergeometric multisums constructed using change of base lemmas for Bailey pairs.

Finally, Bringmann and Kane established two new Bailey pairs and used them to relate certain $q$-hypergeometric series to real quadratic fields. In [51], we show how these pairs give rise to four new mock theta functions in the form of $q$-hypergeometric double sums. For example, if we define
\[
\mathcal{W}(q) := \sum_{n \geq j \geq 1} \frac{(q; q^2)_n(-1)_j(q; q^2)_{j-1}(-1)^{n+j}q^{(j+1)}}{(-q)_n(q)_{n-j}(q)_{2j-1}},
\]
then one of the main results in [51] is the following.

**Theorem 2.9.** $\mathcal{W}(q)$ is a mock theta function.

### 2.4. Mixed mock modular forms.

A recent generalization of mock modular forms is the notion of mixed mock modular forms. These are functions which lie in the tensor space of mock modular forms and modular forms. Mock modular forms in algebra, number theory, and physics are often of the mixed variety. For example, mixed mock modular forms have recently appeared as characters in the theory of affine Lie superalgebras, as generating functions for exact formulas for the Euler numbers of certain moduli spaces, for Joyce invariants and for linking numbers in 3-manifolds, in the quantum theory of black holes and wall-crossing phenomenon, in relation to other automorphic objects and in the combinatorial setting of $q$-series and partitions.

As $q$-series, mixed mock modular forms appear to be much more common than mock theta functions. In [52], we briefly survey some of the ways such series arise. Precisely, we discuss how
q-series transformations, Bailey pairs and the Bailey chain, and partial theta identities naturally lead to mixed mock modular q-series. For example, the multisum

\[ B^{(k)}(q) := \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} q^{n_k^2 + n_{k-1}^2 + \cdots + n_1^2} \]

is a mixed mock modular form for \( k \geq 2 \) [50]. We also give an example of the type of identity one can prove for mixed mock modular forms. To state this identity, recall that on page 9 of the lost notebook [64], Ramanujan recorded what are now known as the tenth order mock theta functions. Two of these, \( \chi \) and \( X \), are defined by

\[ \chi(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q)_{2n+1}} \]

and

\[ X(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q)_{2n}}. \]

**Theorem 2.10.** We have

\[ B^{(2)}(q) + \frac{2}{(q^2, q^3; q^5)_{\infty}} \chi(q) - \frac{2}{(q, q^4; q^5)_{\infty}} X(q) = - \frac{(q)_{\infty}}{(-q)_{\infty}}. \]

2.5. **Quadratic forms.** It is a classical result of Landau from 1908 that the number \( B(x) \) of integers less than or equal to \( x \) which are representable as the sum of two squares \( X^2 + Y^2 \) satisfies the asymptotic formula

\[ B(x) \sim C \frac{x}{\sqrt{\log x}} \quad \text{as } x \to \infty \]  

with the constant

\[ C = \frac{1}{\sqrt{2}} \prod_{p \equiv 3 \ (\text{mod } 4)} \left( \frac{1}{1 - 1/p^2} \right)^{1/2} \approx 0.764223654 \]  

where \( p \) denotes a prime. Let \( f(X, Y) = aX^2 + bXY + cY^2 \) be a primitive positive definite binary quadratic form with discriminant \( D = b^2 - 4ac \) and \( B_f(x) \) be the number of integers less than or equal to \( x \) which are representable by \( f \). Paul Bernays, a doctoral student of Landau’s at Göttingen, proved the following generalization of (2.10) in his 1912 thesis:

\[ B_f(x) \sim C(D) \frac{x}{\sqrt{\log x}} \quad \text{as } x \to \infty \]

with a non-zero constant \( C(D) \) depending only on \( D \). Thus \( C(-4) \) is the constant in (2.11). Bernays did not explicitly give \( C(D) \) for any other value of \( D \). The problem of computing these constants has subsequently attracted considerable attention.

In 1966, Shanks and Schmid [66] studied the family of binary quadratic forms \( f(X, Y) = X^2 + nY^2 \) and used indirect methods to compute \( C(-4n) \) for \( 1 \leq n \leq 14 \) and \( n = 16, 20, 24, 27, 64, 96 \) and 256. They then state: “We note, in passing, that of all binary forms \( u^2 + nv^2 \),
$u^2 + 2v^2$ is the most populous, since $C(-8)$ is the largest of these constants.” It is not completely clear as to whether they meant that $C(-8)$ is the largest amongst the values computed or that the maximum value of $C(-4n)$ as $n$ ranges over all positive integers is assumed for $n = 2$. In any case, this quote motivates the following question: Is $C(-8)$ the maximum value? In [10], we prove the following.

**Theorem 2.11.** If $\Delta$ is a fixed negative fundamental discriminant, then $C(\Delta q)$ is unbounded as $q$ runs through the primes congruent to 1 modulo 4. If $\Delta$ is a fixed positive fundamental discriminant or $1$, then $C(-\Delta q)$ is unbounded as $q$ runs through the primes congruent to 3 modulo 4.

In order to prove Theorem 2.11, we first use a general formula for Bernays’ constant $C(D)$ given in [54]. To our knowledge, this is the first explicit evaluation of $C(D)$ for discriminants $D < 0$. When $D$ is a fundamental discriminant, this formula reduces to

$$C(D) = \frac{1}{2^{\omega(D)-1}} \left[ \frac{|D|}{\varphi(|D|)} \frac{L(1, \chi_D)}{\pi} \prod_{(D/p)=-1} \frac{1}{1-1/p^2} \right]^{1/2} \tag{2.12}$$

where $\omega(D)$ is the number of prime divisors in $D$, $\varphi$ is Euler’s phi function and $L(\cdot, \chi_D)$ is the Dirichlet L-series corresponding to the Kronecker symbol $\chi_D = (D/\cdot)$. We then combine (2.12) with a suitable adjustment of a result due to Joshi [40] to obtain a lower bound for $L(1, \chi_D)$.

2.6. **Automorphic properties of combinatorial generating functions.** Recently, Andrews [2] has extended Dyson’s rank (see Section 2.8 for the definition) to a more general class of partitions. In joint work with Bringmann and Lovejoy [15], we provide a framework which generalizes the work of Andrews in [2] and [3] and determine automorphic properties of the associated $q$-hypergeometric series. First, we give two combinatorial interpretations of the series

$$N_{2v}(d,e; q) := \frac{(-dq, -eq)_\infty}{(q, deq)_\infty} \sum_{n \in \mathbb{Z} \setminus \{0\}} (-1)^{n-1} q^{\binom{n+1}{2} + \nu(n)} (de)^n (-1/d, -1/e)^n (1-q^n)^{2v} (-dq, -eq)_n$$

which recover Andrews’ work [2] when $d = e = 0$. Second, we prove that one specialization of $N_{2v}(d,e; q)$ is a quasimodular form (see Theorem 1.1 in [15]) and three other specializations are quasimock modular functions (see Theorem 1.2 in [15]). These four specializations correspond to all four of the rank functions described below in Section 2.8. The novelty in our approach is that we apply the theory of $q$-series in combination with equations involving partial derivatives in order to deduce automorphic properties. This approach has the advantage in that one typically requires lengthy and delicate analytic calculations [11]. We also mention that these equations involving partial derivatives are of independent interest [5], [14], [16]. Finally, we give a combinatorial interpretation of the series $spt(d,e; q)$ that reduces to Andrews’ [3] $spt$-function when $d = e = 0$. Furthermore, using Theorem 1.2 in [15], we deduce automorphic properties of three specializations of $spt(d,e; q)$ (see Corollary 1.3 in [15]). Theorems 1.1 and 1.2 and Corollary 1.3 provide the theoretical framework necessary to prove any specific number-theoretic fact about these functions.
2.7. Ramanujan-type supercongruences. In Ramanujan’s second letter to Hardy dated February 27, 1913, the following formula appears:

\[
1 - 5\left(\frac{1}{2}\right)^5 + 9\left(\frac{1 \cdot 3}{2 \cdot 4}\right)^5 - 13\left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^5 + \cdots = \frac{2}{\Gamma\left(\frac{3}{4}\right)^4}
\] (2.13)

where \(\Gamma(\cdot)\) is the Gamma function. Note that (2.13) can be expressed as

\[
\sum_{k=0}^{\infty} (4k + 1)\left(\frac{-1}{2}\right)^k = \frac{2}{\Gamma\left(\frac{3}{4}\right)^4}.
\]

In 1996, Van Hamme [69] conjectured 13 \(p\)-adic analogues of similar formulas which relate binomial sums to special values of the Gamma function. For example, he truncated the left-hand side of (2.13) and replaced the Gamma function with the \(p\)-adic Gamma function. Based on numerical computations, he posed the following.

**Conjecture 2.12.** Let \(p\) be an odd prime. Then

\[
\sum_{k=0}^{p-1} (4k + 1)\left(\frac{-1}{2}\right)^k \equiv \begin{cases} -\frac{p}{\Gamma_p\left(\frac{3}{4}\right)^4} \pmod{p^3} & \text{if } p \equiv 1 \pmod{4} \\ 0 \pmod{p^3} & \text{if } p \equiv 3 \pmod{4} \end{cases}
\]

where \(\Gamma_p(\cdot)\) is the \(p\)-adic Gamma function.

In [53], we prove the following.

**Theorem 2.13.** Conjecture 2.12 is true.

The idea of the proof of Theorem 2.13 is as follows. First, one can show that the right-hand side of Conjecture 2.12 is related to a special value of a “Gaussian hypergeometric series”. These series were only recently introduced by Greene in his 1984 Ph.D. thesis. His aim was to show that these series satisfy properties analogous to classical hypergeometric series. For example, evaluations due to Saalschütz, Dixon, Watson, and Whipple all have finite field interpretations. The parameters in these Gaussian series are characters in a finite field. We focus on the case where all of the parameters are characters which are trivial or of order 2. We shall denote this value by \(n_{n+1} F_n(\lambda), \lambda \in \mathbb{F}_p\). Greene showed that \(p^n_{n+1} F_n(\lambda) \in \mathbb{Z}\). Moreover, we have that the right-hand side of Conjecture 2.12 is \(p^3 F_2(1)\). By the main result in [61], we have

\[
p^2 F_2(1) \equiv \left(\frac{-1}{p}\right) \left[p^2 X(p, 1, 2) + pY(p, 1, 2) + Z(p, 1, 2)\right] \pmod{p^3}
\]

where \(X(p, \lambda, n), Y(p, \lambda, n)\) and \(Z(p, \lambda, n)\) are explicitly given in terms of binomial sums. We then show that \(Y(p, 1, 2) \equiv X(p, 1, 2) \equiv 0 \pmod{p}\). Finally, one uses Whipple’s transformation (for ordinary hypergeometric series) to prove that the left-hand side of Conjecture 2.12 is congruent to \(\left(\frac{-1}{p}\right)pZ(p, 1, 2) \pmod{p^3}\).
2.8. Rank differences. A partition of a non-negative integer \( n \) is a non-increasing sequence of positive integers whose sum is \( n \). Let \( p(n) \) denote the number of partitions of \( n \). The rank of a partition is the largest part minus the number of parts. This statistic was introduced by Dyson [23], who observed empirically that it provided a combinatorial explanation for Ramanujan’s congruences \( p(5n + 4) \equiv 0 \pmod{5} \) and \( p(7n + 5) \equiv 0 \pmod{7} \). Specifically, Dyson conjectured that if \( N(s, m, n) \) denotes the number of partitions of \( n \) whose rank is congruent to \( s \) modulo \( m \), then for all \( 0 \leq s \leq 4 \) and \( 0 \leq t \leq 6 \) we have

\[
N(s, 5, 5n + 4) = \frac{p(5n + 4)}{5}
\]

and

\[
N(t, 7, 7n + 5) = \frac{p(7n + 5)}{7}.
\]

Atkin and Swinnerton-Dyer proved these assertions in 1954 [6]. In fact, they proved much more, establishing generating functions for every rank difference

\[
N(s, \ell, \ell n + d) - N(t, \ell, \ell n + d)
\]

with \( \ell = 5 \) or \( 7 \) and \( 0 \leq d, s, t < \ell \).

The rank of a partition studied by Dyson, Atkin and Swinnerton-Dyer is now understood to be a special case of a more general rank which is defined on overpartition pairs [13]. Recall that an overpartition of \( n \) is a partition of \( n \) where we may overline the first occurrence of a part while an overpartition pair \((\lambda, \mu)\) of \( n \) is a pair of overpartitions where the sum of all of the parts is \( n \). Overpartitions and overpartition pairs naturally arise in diverse areas of mathematics where partitions already occur, such as mathematical physics, symmetric functions, representation theory and algebraic number theory. The rank of an overpartition pair \((\lambda, \mu)\) is

\[
\ell((\lambda, \mu)) - n(\lambda) - \pi(\mu) - \chi((\lambda, \mu))
\]

where \( \pi(\cdot) \) is the number of overlined parts only and \( \chi((\lambda, \mu)) \) is defined to be 1 if the largest part of \((\lambda, \mu)\) occurs only non-overlined and only in \( \mu \), and 0 otherwise. When \( \mu \) is empty and \( \lambda \) has no overlined parts, (2.14) becomes Dyson’s rank of a partition. In addition to this rank, three other special cases of (2.14) have turned out to be of particular interest: Dyson’s rank of an overpartition, the \( M_2 \)-rank of a partition without repeated odd parts and the \( M_2 \)-rank of an overpartition.

In joint work with Lovejoy [47], [48], we apply the method of Atkin and Swinnerton-Dyer to find formulas for the generating functions for rank differences for overpartitions and \( M_2 \)-rank differences for partitions without repeated odd parts. In [49], we complete the picture by doing the same for \( M_2 \)-rank differences for overpartitions. As an example of one of our main results, we have the following. Let \( \overline{N}(s, m, n) \) be the number of overpartitions of \( n \) with Dyson’s rank congruent to \( s \) modulo \( m \) and

\[
R_{st}(d) := \sum_{n=0}^{\infty} \left( \overline{N}(s, \ell, \ell n + d) - \overline{N}(t, \ell, \ell n + d) \right) q^n.
\]

In [47], we prove the following.

**Theorem 2.14.** For \( \ell = 3 \), we have
Theorem 2.15. For the fields \(\mathbb{Q}(\sqrt{p_1l})\) and \(\mathbb{Q}(\sqrt{2p_1l})\), 4-rank 1 and 2 each appear with natural density \(\frac{1}{7}\) in \(\Omega\). For the fields \(\mathbb{Q}(\sqrt{-p_1l})\) and \(\mathbb{Q}(\sqrt{-2p_1l})\), 4-rank 0 and 1 each appear with natural density \(\frac{1}{2}\) in \(\Omega\).

The idea behind Theorem 2.15 is to use the matrices of local Hilbert symbols to get a correspondence between 4-ranks values and characterizations of the primes \(p\) and \(l\) by positive definite binary quadratic forms. This characterization then determines the splitting of \(p\) and \(l\) in a certain normal extension of \(\mathbb{Q}\). Associating Artin symbols to \(p\) and \(l\), we then use the Cebotarev density theorem. In [56], we extend the results in [59] by providing a complete density picture for primes \(p\) binary quadratic forms. This characterization then determines the splitting of \(p\) and \(l\) in a certain normal extension of \(\mathbb{Q}\).
for the 4-ranks of tame kernels of the fields $\mathbb{Q}(\sqrt{pl})$, $\mathbb{Q}(\sqrt{-pl})$ for primes $p$, $l$. In [60], we discuss a relationship between the matrices of local Hilbert symbols and Rédei matrices [65] which were used in the 1930’s to study the structure of ideal class groups.

3. Selected Current Projects

3.1. $q$-series identities from knot theory. The $q$-multisum $\Phi_K(q)$ from the Rogers-Ramanujan type identities in Section 2.1 occurs as the 0-limit (or “tail”) of the colored Jones polynomial of $K$. We will study $q$-series identities in other settings which arise from knot theory. First, Garoufalidis and Lê have obtained an explicit formula (see Theorem 1.14 in [27]) for the 1-limit (or “tail of the tail”) of the colored Jones polynomial of $K$. Second, we will investigate the existence of “tails” for two generalizations of the colored Jones polynomials, namely the two-variable colored HOMFLY polynomial and the three-variable colored superpolynomials. Identities for the $(2, 2p + 1)$ torus knots [26] and twist knots [57] suggest that a multitude of such expressions should exist.

3.2. New mock theta functions and WRT invariants. There is an alluring connection between mock theta functions and invariants from topological quantum field theory. In 1999, Lawrence and Zagier [45] showed that Ramanujan’s 5th order mock theta functions coincide asymptotically with Witten-Reshetikhin-Turaev (WRT) invariants of Poincaré homology spheres (see also [35]). A unified WRT invariant for integral homology spheres was recently proposed by Habiro [30]. Interestingly, the computation of this unified WRT invariant for certain manifolds actually leads to new mock theta functions. For example, if we define the $q$-series

$$\phi(q) := \sum_{n=0}^{\infty} q^n(-q)_{2n+1},$$

then the main result in [12] says that $\phi(q)$ is a mock theta function and $\phi(-q^{1/2})$ is (up to an explicit factor) the unified WRT invariant of the Seifert manifold $\Sigma(2,3,8)$ which arises from +2 surgery on the trefoil knot. More generally, it is known [17] that +2 surgery on the twist knot $K_p$ gives a Seifert manifold with unified WRT invariant

$$B_p(q) = \sum_{s_p \geq \cdots \geq s_1 \geq 0} (-1)^{s_p} q^{s_p/2}(q^{1/2}; -q^{1/2})_{2s_p+1} \prod_{i=1}^{p-1} q^{s_i(s_i+1)} \frac{(q)_{s_{i+1}}}{(q)_{s_{i+1}} - (q)_{s_i}}.$$

Note that for $p = 1$, $K_1$ is the trefoil knot and $B_1(q) = \phi(-q^{1/2})$. The modularity of $B_p(q)$ for $p \geq 2$ is not currently known. The goal of this project is to compute many more unified WRT invariants of manifolds which arise from various surgeries on other knots and apply the vast theory of $q$-series identities in order to discover new mock theta functions.

3.3. $q$-hypergeometric series and black holes. Very recently, Dabholkar, Murty and Zagier [22] have shown that the meromorphic Jacobi form that counts the quarter-BPS states in $\mathcal{N} = 4$ string theories can be expressed as a sum of a mock Jacobi form (whose Fourier coefficients are the quantum degeneracies of single-centered black holes) and an Appell-Lerch sum. Moreover, they study two classes of related functions which not only recover the mock theta functions of Ramanujan, but also the generating function of Hurwitz-Kronecker class numbers and the mock modular forms appearing in the Mathieu and Umbral moonshine. The authors then state “we
should mention that all of Ramanujan’s [mock theta] functions were \( q \)-hypergeometric series but that in general no \( q \)-hypergeometric expression for the theta expansion coefficients of the [related] functions is known. This seems to be an interesting subject for further research”. The goal of this project is to use the data given in the appendices of [22] plus a computational technique developed by Andrews called the Engel expansion of a power series to first determine a conjectural \( q \)-hypergeometric series representation of these functions. We will then use \( q \)-series techniques to prove the conjectured equality.

3.4. Embedding partial theta identities. In Ramanujan’s lost notebook, the following “marvelous” \( q \)-series transformation appears:

\[
\sum_{n=0}^{\infty} (a)_{n+1} (q/a)_n q^n = \sum_{n=0}^{\infty} (-1)^n a^{3n} q^{n(n+1)/2} (1 - a^2 q^{2n+1}) + \frac{1}{(a)_\infty (q/a)_\infty} \sum_{n=0}^{\infty} (-1)^n a^{2n+1} q^{n(n+1)/2}.
\] (3.1)

This is one example of a partial theta identity, so-called as it contains the partial theta product \((a)_{n+1} (q/a)_n\) and the partial theta sum \(\sum_{n=0}^{\infty} (-1)^n a^{2n} q^{n(n+1)/2}\). In [70], Warnaar shows how (3.1) and three other partial theta identities can be embedded into an infinite family of identities. For example, the left-hand side becomes a \(k\)-dimensional multisum and the second term on the right-hand side becomes the sum of a finite number of infinite series. The \(k = 1\) case then recovers (3.1). Curiously, there are other partial theta identities of Ramanujan which escape this unification. For example,

\[
\sum_{n=0}^{\infty} (a; q^3)_{n+1} (q^2/a; q^3)_n q^{3n^2} - q \sum_{n=1}^{\infty} \frac{q^{3n-1}}{(aq^2, q^2/a; q^3)_n} + \frac{q}{a} \sum_{n=1}^{\infty} \frac{q^{3n-1}}{(aq, q^2/a; q^3)_n} = \frac{(q^2)_\infty}{(q^3)_\infty (a, q/a)_\infty}.
\] (3.2)

The goal of this project is to use well-chosen Bailey pairs and the Bailey chain to embed not only (3.2), but other mysterious partial theta identities into infinite families of identities.

3.5. Character sums. For a prime \(p \equiv 3 \pmod{4}\), let

\[S(k) := \sum_{n=1}^{p-1} \left( \frac{n}{p} \right) n^k.\]

Ayoub, Chowla and Walum [7] noted that \(S(0) = 0, S(1) < 0\) and \(S(2) < 0\). They also proved that when \(k = 3\) there are infinitely many primes \(p \equiv 3 \pmod{4}\) for which \(S(k) > 0\) and infinitely many for which \(S(k) < 0\). Unfortunately, their method does not work for all \(k\). Fine [24] proved that for each real \(k > 2\) there are infinitely many primes \(p \equiv 3 \pmod{4}\) for which \(S(k) > 0\) and infinitely many for which \(S(k) < 0\). In 1979, Cook [19] obtained a quantitative version of Fine’s result. The key to Cook’s result is to first express \(S(k)\) in terms of \(L(1, \chi_p)\) and then apply an improvement of Joshi’s result [40]. One possible extension is as follows. For a squarefree positive integer \(D\), consider the character sum
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$$S(k, \chi_D) := \sum_{n=1}^{D-1} \chi_D(n) n^k$$

where $\chi_D = (D/ \cdot )$. We will study $S(k, \chi_D)$ in order to obtain quantitative results concerning the number of $D$’s for which $S(k, \chi_D) > 0$ and the number of $D$’s for which $S(k, \chi_D) < 0$.

3.6. Vandiver’s conjecture and the vanishing of eigenspaces. Let $p > 3$ be a prime, $\zeta_p$ a $p$th primitive root of unity, $\Delta$ the Galois group of $\mathbb{Q}(\zeta_p)$ over $\mathbb{Q}$ and $A$ the $p$-Sylow subgroup of the ideal class group of $\mathbb{Q}(\zeta_p)$. Vandiver’s conjecture is the statement that $p$ does not divide the class number of the field $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$. One can reformulate this conjecture to say that eigenspaces $e_r$ which occur in the decomposition of $A$ as a $\mathbb{Z}_p[\Delta]$-module vanish for all even $r$, $2 \leq r \leq p - 3$. Kurihara [44] used the surjectivity of the Chern map

$$K_4(\mathbb{Z}) \otimes \mathbb{Z}/p \rightarrow e_{p-3}$$

in order to prove that the “top” even eigenspace $e_{p-3}$ always vanishes. There is an approach due to Thaine [68] which provides an alternative way to prove the vanishing of eigenspaces. For $1 \leq n \leq p - 2$, consider the Jacobi sum $J_n$ written as

$$J_n = \sum_{k=0}^{p-1} d_{n,k} \zeta_p^k$$

where $d_{n,k}$ are integers such that $\sum_{k=0}^{p-1} d_{n,k} = 1$. The main result in [68] gives a criterion to recognize, in terms of the coefficients $d_{n,k}$, if Vandiver’s conjecture holds. Namely, if we consider a certain double sum involving the $d_{n,k}$ and show that it is not zero modulo $p$, then $e_r$ is trivial for all even $r$, $2 \leq r \leq p - 3$. In fact, Thaine shows that in order to study a given component $e_r$, one needs the coefficients $d_{n,k}$ for only one convenient integer $n$.

We will investigate two applications of this result. The first is to give an alternative proof of Kurihara’s result on the vanishing of $e_{p-3}$. This corresponds to taking $n = 1$ in $d_{n,k}$ and, by Proposition 7 in [68], proving that

$$\sum_{k=1}^{p-1} \sum_{l=1}^{p-1} k l^2 d_{1,k} d_{1,k+l} \not\equiv 0 \pmod{p} \quad (3.3)$$

for all odd primes $p \geq 5$. The second is to prove that if 2 is a primitive root modulo $p$, then $e_r$ is trivial for all even $r$, $2 \leq r \leq p - 3$. This case also corresponds to taking $n = 1$. The idea would then be to prove that the double sum in (3.3) is not 0 modulo $p$ where $l^2$ is replaced with $p^{-1-r}$.

3.7. The “1/3 density conjecture”. There is a beautiful question about the norm of the fundamental unit in a real quadratic field [39]. Let $p$ be a prime and consider the quadratic field $K = \mathbb{Q}(\sqrt{2p})$ with fundamental unit $\epsilon > 1$. If $p \equiv 3 \pmod{4}$, then the norm of $\epsilon$ is 1. Now consider the set

$$S = \{p \equiv 1 \pmod{4} : N_{K/\mathbb{Q}}(\epsilon) = -1\}.$$
Numerical computations suggest the following question: Does $S$ have natural density $1/3$ in the set of all primes? More generally, one can consider the set $D$ of discriminants of real quadratic fields that are not divisible by any prime congruent to 3 modulo 4. Now let $D^- \subset D$ be the set of discriminants $D$ of real quadratic fields for which the fundamental unit $\epsilon$ has norm $-1$. Stevenhagen [67] has conjectured that the density of $D^-$ in $D$ exists and equals

$$1 - \prod_{j \geq 1 \text{ odd}} (1 - 2^{-j}) = \ldots$$

There has been recent spectacular work of Fouvry and Klüners [25] in this direction. We plan to apply their methods to the restricted setting of $K = \mathbb{Q}(\sqrt{2p})$ in order to settle the “1/3 density conjecture”.

3.8. An algorithm for computing the 2-part of $K_2(\mathcal{O}_F)$. Let $F = \mathbb{Q}(\sqrt{d})$ be a quadratic number field with discriminant $D$. It is a classical problem in number theory to study the structure of the narrow ideal class group $Cl^+(F)$, in particular the 2-part. Indeed, Gauss showed that 2-rank $Cl^+(F) = r - 1$ where $r$ is the number of distinct prime divisors of $D$. In 1934, L. Rédei [65] gave a formula for the 4-rank of $Cl^+(F)$ and in 1973, W. Waterhouse [71] gave a method for computing the 8-rank of $Cl^+(F)$. The purpose of the paper [43] is to provide an algorithm to compute all of the $2^k$-ranks, $k \geq 2$, of $Cl^+(F)$ for arbitrary quadratic fields $F$, thereby generalizing the methods of Rédei and Waterhouse. For real quadratic fields, the algorithm detects the norm of the fundamental unit and the structure of the 2-part of the ordinary class group $Cl(F)$ of $F$ in the cases where the norm of the fundamental unit is 1.

Kolster’s algorithm works as follows: For $F = \mathbb{Q}(\sqrt{d})$ with $r$ ramified primes $p_1, p_2, \ldots, p_r$, construct “Rédei matrices” $R^{(k)}(F)$, $k \geq 1$, of size $r \times r$ and rank $r_k \leq r - 1$. The entries in $R^{(k)}(F)$ are local Hilbert symbols $(a_{j,k}, d)_{p_i}$, where $a_{j,k}$ are certain integers. The main result in [43] says

$$2^{k+1} \text{-rank } Cl^+(F) = r - 1 - r_k$$

for $k \geq 1$. If $r_k < r - 1$, then the matrix $R^{(k+1)}(F)$ is obtained from $R^{(k)}(F)$ by updating the entries in the $r - r_k$ dependent rows of $R^{(k)}(F)$ using solutions to certain norm equations. We plan to use the fact that the matrices of local Hilbert symbols (mentioned in Section 2.9) are analogous to Rédei matrices in order to adapt Kolster’s algorithm to determine the 2-part of $K_2(\mathcal{O}_F)$.

3.9. $K$-groups and combinatorics. Consider the narrow ideal class groups $Cl^+(F)$ and $Cl^+(K)$ for the quadratic number fields $F = \mathbb{Q}(\sqrt{d})$ and $K = \mathbb{Q}(\sqrt{-d})$. A famous Spiegelungssatz (“reflection theorem”) tells us that the 4-ranks of $Cl^+(F)$ and $Cl^+(K)$ differ by at most 1. Precisely, we have

$$4 \text{-rank } Cl^+(F) \leq 4 \text{-rank } Cl^+(K) \leq 4 \text{-rank } Cl^+(F) + 1. \quad (3.4)$$

Recently, Habsieger and Royer [31] gave a combinatorial proof of (3.4). The goal of this project is to first consider the analogue of (3.4) for the 4-ranks of $K_2(\mathcal{O}_F)$ and $K_2(\mathcal{O}_K)$, then give a combinatorial proof of this $K$-theoretic Spiegelungssatz.
References


