Cyclotomic Units and Greenberg's conjecture

For an abelian number field F, already Kummer (for $F = \mathbb{Q}(\zeta_n)^+$), and eventually Sinnott and others, defined the so-called *cyclotomic units* as a subgroup of full rank of the usual units $\mathcal{O}_F^\times$ with the striking property that their index in the full group of units coincides - up to known explicit constants - with the class number of F. Let p be a prime not dividing $]F : \mathbb{Q}]$. Then Sinnot proved that these explicit constants stay bounded along the cyclotomic p-extension of F and they are prime to p: in other words, at every level F_n of the cyclotomic extension the p-part of the index of the cyclotomic units in $\mathcal{O}_{F_n}^\times$ equals the order of the p-Sylow of the class group Cl_{F_n}.
It is then natural to ask whether this equality comes from an isomorphism (of abelian groups, say). Works by Kraft−Schoof and Kuz'min, later refined by Ozaki and by Belliard−Nguyen-Quang-Do, shows that if p does not split in F then a famous conjecture by Greenberg would indeed imply that the p-Sylow of Cl_{F_n} and of the quotient of $\mathcal{O}_{F_n}^\times$ by the cyclotomic units are eventually isomorphic as Galois modules. In my talk I would like to discuss the case when p splits in F, showing that Greenberg's conjecture implies that the natural map, giving the isomorphism in the non-split case, has non-trivial kernels and cokernels.