

PRINCIPAL FORMS $X^2 + nY^2$ REPRESENTING MANY INTEGERS

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ABSTRACT. In 1966, Shanks and Schmid investigated the asymptotic behavior of the number of positive integers less than or equal to x which are represented by the quadratic form $X^2 + nY^2$. Based on some numerical computations, they observed that the constant occurring in the main term appears to be the largest for $n = 2$. In this paper, we prove that in fact this constant is unbounded as n runs through positive integers with a fixed number of prime divisors.

1. INTRODUCTION

It is a classical result of Landau [14] from 1908 that the number $B(x)$ of integers less than or equal to x which are representable as the sum of two squares $X^2 + Y^2$ satisfies the asymptotic formula

$$(1) \quad B(x) \sim C \frac{x}{\sqrt{\log x}} \quad \text{as } x \rightarrow \infty$$

with the constant

$$C = \frac{1}{\sqrt{2}} \prod_{p \equiv 3 \pmod{4}} \left(\frac{1}{1 - 1/p^2} \right)^{1/2} = 0.764223654,$$

where p denotes a prime. Independently, Ramanujan in his first letter to Hardy in 1913 stated essentially that

$$B(x) \sim C \int_2^x \frac{dt}{\sqrt{\log t}} \quad \text{as } x \rightarrow \infty.$$

Later claims by Hardy [9] that Ramanujan's integral did not give a better estimate of $B(x)$ than Landau's simpler formula were shown to be false by Shanks [17]. The constant C is now called the *Landau-Ramanujan constant*.

Consider a primitive quadratic form $f(X, Y) = aX^2 + bXY + cY^2$ over \mathbb{Z} with non-square discriminant $D = b^2 - 4ac$, and suppose f is positive in case it is definite. Let $B_f(x)$ be the number of positive integers less than or equal to x which are representable by f . Paul Bernays, a doctoral student of Landau's at Göttingen, proved the following generalization of (1) in his 1912 thesis [2, pages 59 and 115–116]:

$$(2) \quad B_f(x) \sim C(D) \frac{x}{\sqrt{\log x}} \quad \text{as } x \rightarrow \infty$$

with a non-zero constant $C(D)$ depending only on D . Thus $C(-4)$ is the Landau-Ramanujan constant. Bernays did not explicitly give $C(D)$ for any other value of D . The problem of computing these constants has subsequently attracted considerable attention.

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The original method of Landau can be used to compute $C(D)$ when the class number $h(D)$ is 1 or, with some additional complications, not too big. In 1966, Shanks and Schmid [18] studied the forms $f(X, Y) = X^2 + nY^2$ and determined the corresponding constants $C(-4n)$ (b_n in their notation) in this way for 30 values of n in the range $-34 \leq n \leq 256$ with class numbers $h(-4n) = 1, 2, 3, 4$, and 8. In particular they find

$$C(-8) = 0.872887558.$$

They then state (page 561) “We note, in passing, that of all binary forms $u^2 + nv^2$, $u^2 + 2v^2$ is the most populous, since b_2 is the largest of these constants.” It is not completely clear as to whether they meant that $C(-8)$ is the largest amongst the values computed or that the maximum value of $C(-4n)$ as n ranges over all integers is assumed for $n = 2$. In any case, this quote motivates the following question: Is $C(-8)$ the maximum value? The purpose of the present paper is to answer this question. Specifically we prove:

Theorem 1.1 *If Δ is a fixed negative fundamental discriminant, then $C(\Delta q)$ is unbounded as q runs through the primes congruent to 1 modulo 4. If Δ is a fixed positive fundamental discriminant or 1, then $C(-\Delta q)$ is unbounded as q runs through the primes congruent to 3 modulo 4.*

It follows for example that $C(-4q)$ is unbounded as q runs through the primes congruent to 1 modulo 4. However, it is not easy finding a concrete such q with $C(-4q) > C(-8)$. We have been able to find only one such example, namely

$$(3) \quad C(-4 \cdot 13779962790518414129) = 0.875986.$$

In Section 2 we present a formula for $C(D)$ in the case D is a fundamental discriminant and sketch how it is derived. In Section 3 we discuss computational aspects of this formula. In Section 4 we prove Theorem 1.1 by making some adjustments in a proof of Joshi [12]. In the final section we formulate some further problems and questions related to $C(D)$.

2. MAKING $C(D)$ EXPLICIT

Using results of Kaplan and Williams [13] and Sun and Williams [19], an explicit formula for Bernays’ constant $C(D)$ for discriminants $D < 0$ was given in [16] (see (2.5), (2.8), and (2.11)). When D is a fundamental discriminant, i.e., equal to the discriminant of a quadratic number field, this formula reduces to

$$(4) \quad C(D) = \frac{1}{2^{\omega(D)-1}} \left[\frac{|D|}{\varphi(|D|)} \frac{L(1, \chi_D)}{\pi} E(D) \right]^{1/2}.$$

Here $\omega(D)$ is the number of prime divisors in D , φ is Euler’s phi function, $L(\cdot, \chi_D)$ is the Dirichlet L-series corresponding to the Kronecker symbol $\chi_D = (D/\cdot)$, and $E(D)$ is the Euler product

$$E(D) = \prod_{(D/p)=-1} \frac{1}{1 - 1/p^2}.$$

Note that

$$1 < E(D) < \prod_p \frac{1}{1 - 1/p^2} = \frac{\pi^2}{6}$$

and thus the contribution of $E(D)$ to $C(D)$ is limited.

We will now sketch some arguments that go into the derivation of (4). Recall that a primitive positive definite quadratic form $aX^2 + bXY + cY^2$, $[a, b, c]$ for short, is said to be *reduced* if $|b| \leq a \leq c$, and $b \geq 0$ if either $|b| = a$ or $a = c$. Every primitive positive definite form is properly equivalent to a unique reduced form [5, Theorem 2.8]. This reduced form represents precisely the same integers as the original one. Thus we might assume from the outset that our form is reduced. We say that two forms are in the same *class* if they are properly equivalent. It is easy to see that the number $h(D)$ of classes of primitive positive definite forms of discriminant D is finite, and furthermore $h(D)$ is equal to the number of reduced forms of discriminant D [5, Theorem 2.13]. In the case D is a fundamental discriminant, $h(D)$ also equals the class number of the quadratic number field $\mathbb{Q}(\sqrt{D})$, and we have

$$\frac{L(1, \chi_D)}{\pi} = u(D) \cdot \frac{h(D)}{\sqrt{-D}}$$

where $u(-3) = 1/3$, $u(-4) = 1/2$, and $u(D) = 1$ for $D < -4$.

We say that two primitive positive definite forms of discriminant D are in the same *genus* if they represent the same values in $(\mathbb{Z}/D\mathbb{Z})^*$. The number of genera is known to be a power of two and so can be written as $2^{t(D)}$ for some $t(D) \geq 0$. The number $t(D)$ depends only on $\omega(D)$ and the congruence class of D modulo 32 (see (2.7) in [16]). Furthermore, given a negative integer $D \equiv 0, 1 \pmod{4}$, the *principal form* is defined by $[1, 0, -\frac{D}{4}]$ if $D \equiv 0 \pmod{4}$ and $[1, 1, \frac{1-D}{4}]$ if $D \equiv 1 \pmod{4}$. The principal form has discriminant D and is reduced. If $D = -4n$, then $[1, 0, n]$ is the principal form. Finally, we say that an integer m is represented by the genus G if it is represented by at least one class in G . For example, if $n = 14$ (and thus $D = -56$ and $h(D) = 4$), then we have two genera G_1 and G_2 where, say, (the class of) $[1, 0, 14]$ and $[2, 0, 7]$ belong to G_1 while $[3, -2, 5]$ and $[3, 2, 5]$ belong to G_2 .

Let us first consider the simpler problem of deriving an analogue of (2) with $B'_f(x)$ instead of $B_f(x)$, where $B'_f(x)$ counts the number of integers $m \leq x$ represented by f with $(m, D) = 1$. Note that without loss of generality we may assume that f is reduced. Let f belong to the genus G . The counting strategy is as follows:

- (1) Compute $B'_D(x)$; the number of integers $m \leq x$ coprime to D that are represented by *any* reduced form of discriminant D .
- (2) Compute $B'_G(x)$; the number of integers $m \leq x$ coprime to D that are represented by G .
- (3) Compute $B'_f(x)$,

where the result of each step provides input for the next. We now proceed through these steps.

(1) An integer $m \leq x$ is counted if and only if its prime divisors that occur to an odd power satisfy $(D/p) = 1$. The associated L-series $L_D(s) = \sum m^{-s}$ thus has, for $\text{Re}(s) > 1$, the following

Euler product:

$$L_D(s) = \prod_{(D/p)=1} (1-p^{-s})^{-1} \prod_{(D/p)=-1} (1-p^{-2s})^{-1}.$$

Using the Euler product formula for $\zeta(s)$ and $L(s, \chi_D)$ one finds, for $\operatorname{Re}(s) > 1$, that

$$L_D(s)^2 = \zeta(s)L(s, \chi_D) \prod_{p|D} (1-p^{-s}) \prod_{(D/p)=-1} (1-p^{-2s})^{-1} = \zeta(s)g(s),$$

say. Then using the Selberg-Delange method (see, e.g., [6, Proposition 5]), one finds that

$$B'_D(x) \sim \sqrt{\frac{g(1)}{\pi}} \frac{x}{\sqrt{\log x}},$$

where

$$\sqrt{\frac{g(1)}{\pi}} = \left[\frac{\varphi(|D|)}{|D|} \frac{L(1, \chi_D)}{\pi} E(D) \right]^{1/2} =: J(D),$$

the James constant, who first established this result [11] (from a modern perspective this result is completely standard). For different proofs see [16, 20]. The $\sqrt{\log x}$ is due to the fact that asymptotically half of the primes are represented by a reduced binary quadratic form. If we have many small primes p with $(D/p) = 1$, then $L(1, \chi_D)$ is large on the one hand, and on the other many integers m are counted by $B'_D(x)$. Thus it is natural to expect that $J(D)$ scales as an increasing function of $L(1, \chi_D)$.

(2) Any integer coprime to D is represented by at most one genus. Bernays showed that, as x tends to infinity, the integers $m \leq x$ become equidistributed over the genera, that is, we have

$$B'_G(x) \sim \frac{B_D(x)}{2^{t(D)}} \sim \frac{J(D)}{2^{t(D)}} \frac{x}{\sqrt{\log x}}.$$

(3) We let E be the set of integers that are represented by G , but not by f , and $E(x)$ the associated counting function (for example, in the above example 2 and 7 are represented by G_1 , but not by [1, 0, 14]). Bernays showed that $E(x) = o(x \log^{-1/2} x)$ (this result was later sharpened by Fomenko [7] to $E(x) \ll x \log^{-2/3} x$). We finally conclude that

$$B'_f(x) \sim B'_G(x) \sim \frac{J(D)}{2^{t(D)}} \frac{x}{\sqrt{\log x}}.$$

This solves the asymptotic counting problem for $B'_f(x)$. However, we are interested in $B_f(x)$. A complication that arises here is that more than one genus might represent an integer not coprime to D . Bernays took this complication into account and arrived at the following formula

$$C(D) = \frac{J(D)}{2^{t(D)}} \sum_{m|D^\infty} \frac{g(m, D)}{m},$$

where $g(m, D)$ denotes the number of genera of discriminant D representing m and $m|D^\infty$ means that m divides some arbitrary power of D . The latter sum is a rational number and its explicit evaluation was only made possible by the recent papers [13, 19] mentioned in the beginning of this section. In the case D is a fundamental discriminant the latter sum equals $|D|/\varphi(|D|)$, $t(D) = \omega(D) - 1$ and we obtain (4). For the general formula we refer to [16].

Remark 1. In retrospect one sees why the classical Landau case readily follows. There one has $B'_{-4}(x) = B'_G(x) = B'_f(x)$. Let $Od(x)$ denote the number of odd integers $\leq x$ that can be written as a sum of two squares. Thus $B'_{-4}(x) = Od(x)$. Note that $J(-4) = C/2$ and $B(x) = \sum_{j=0}^{\infty} Od(x2^{-j})$. We infer that $B(x) \sim 2Od(x) \sim Cx \log^{-1/2} x$. Observe that $h(-4) = 1$. Indeed, the three cases coincide if and only if $h(D) = 1$. The second and third case coincide (that is every genus consists of one class) if and only if $n \geq 1$ is a convenient number (“numerus idoneus”). This happens for at least 65 and at most 66 integers [5, pp. 61-62].

3. COMPUTATION OF $C(D)$

The class numbers $h(D)$ can be computed for about $|D| < 10^{20}$ by GP/PARI. In order to compute $C(D)$ accurately, the problem thus lies in computing the Euler product $E(D)$. Taking the product over, say, the first 100,000 primes gives a precision of about 6 decimals. A much better precision is obtained by the rapidly converging infinite product (see [17, eq. (16)] or [16, eq. (3.3)])

$$E(D) = \prod_{k=1}^{\infty} \left(\frac{\zeta(2^k)}{L(2^k, \chi_D)} \prod_{p|D} (1 - p^{-2^k}) \right)^{1/2^k}.$$

However, the values $L(2^k, \chi_D)$ can be computed only for about $|D| < 10^6$ by GP/PARI.

Using the above formulas, we compute $C(D)$ for some small fundamental discriminants $D < 0$ ordered according as to whether $\omega(D)$ is 1, 2, 3, or 4:

D	$C(D)$	D	$C(D)$	D	$C(D)$	D	$C(D)$
-3	0.638909405	-15	0.501918636	-84	0.310647641	-420	0.164080141
-4	0.764223654	-20	0.535179999	-120	0.296417662	-660	0.143806822
-7	0.724719521	-24	0.558357114	-132	0.274765289	-840	0.139069358
-8	0.872887558	-35	0.407379938	-168	0.267006498	-1092	0.123274604
-11	0.677388018	-39	0.518747305	-195	0.220993565	-1140	0.171607125
-19	0.606300131	-40	0.473558100	-228	0.237562625	-1155	0.109195133
-23	0.841512352	-51	0.390646647	-231	0.309699577	-1320	0.121504603
-31	0.801014576	-52	0.420720518	-255	0.307681243	-1380	0.117420083
-43	0.500610055	-55	0.458949554	-260	0.293752522	-1428	0.114424422
-47	0.891550880	-56	0.563486772	-264	0.319941656	-1540	0.108139197
-59	0.735485997	-68	0.520288297	-276	0.309309571	-1560	0.161366493
-67	0.448813095	-87	0.512573818	-280	0.223644570	-1716	0.148895032
-71	0.938541302	-88	0.375792661	-308	0.277034255	-1848	0.109066658
-79	0.812629337	-91	0.317487516	-312	0.223049066	-1860	0.151207258
-83	0.684502354	-95	0.528624390	-340	0.204812008	-1995	0.093833104

It appears that $2^{1-\omega(D)}$ tends to dominate the other factors in (4). We have $\omega(D) = 1$ when D is of the form $-q$ for a prime $q \equiv 3 \pmod{4}$. It is straightforward to find such primes with $C(-q) > C(-8)$, for example $q = 47, 71, 167, 191$ or 239 . The largest value of $C(-q)$ with $q < 10^9$, which is also the largest value of $C(D)$ that we know, is

$$C(-984452999) = 1.527855.$$

But already for $\omega(D) = 2$ it becomes much more difficult to find a D with $C(D) > C(-8)$, and the only example we know is (3). The difficulty in finding such examples is explained by the fact that

$$\max_{0 < -D < x} L(1, \chi_D)$$

grows very slowly with x . For example, Bateman, Erdős, and Chowla [1] proved

$$L(1, \chi_D) < \frac{10}{3} \frac{\varphi(|D|)}{|D|} \log |D| + 1.$$

Moreover, assuming a suitable generalized Riemann hypothesis, Littlewood [15] showed

$$(5) \quad e^\gamma \leq \limsup_{D \rightarrow -\infty} \frac{L(1, \chi_D)}{\log \log |D|} \leq 2e^\gamma$$

with D running through negative fundamental discriminants. The left-hand inequality in (5) was shown unconditionally by Chowla [3] (see also the discussion in [1]). Recent work by Granville and Soundararajan [8] gives strong evidence via a probabilistic model that e^γ is in fact the true limit superior of $L(1, \chi_D)/\log \log |D|$.

Regarding small values of $C(D)$, we mention that it was shown in [16] that $C(D) \cdot \sqrt{-D}$ is minimal for $D = -3$ as D ranges over all negative discriminants, and that accordingly of all the two-dimensional lattices of covolume 1, the hexagonal lattice has asymptotically the fewest distances.

4. PROOF OF THEOREM 1.1

In order to prove Theorem 1.1, we need to show that $L(1, \chi_D)$ is unbounded when D runs through a certain subset of discriminants. In this direction, it is proved in [1] that

$$(6) \quad \limsup_{D \rightarrow -\infty} \frac{L(1, \chi_D)}{\log \log |D|} \geq \frac{e^\gamma}{18}$$

where D runs through fundamental discriminants of the form $D = -q$ with $q \equiv 3 \pmod{4}$ prime. This implies that $C(D)$ is unbounded, but says nothing about discriminants of the form $D = -4n$. Our main interest is in a result of Joshi [12] in which she improved (6) by removing the factor 18 (for a quantitative version of this result, see [4]). It turns out that one can make suitable adjustments to Joshi's proof in order to prove the following:

Theorem 3.1. *Let Δ be a fundamental discriminant or 1, let c and d be coprime integers with d divisible by Δ and 8, let q run through the primes congruent to $c \pmod{d}$, and let χ be the Kronecker character $(\Delta q^*/\cdot)$ with $q^* = \lambda q$, $\lambda = (-1)^{(c-1)/2}$. Then*

$$\limsup_{\substack{q \rightarrow \infty \\ q \equiv c \pmod{d}}} \frac{L(1, \chi)}{\log \log q} \geq e^\gamma \cdot \prod_{p|d} \frac{1 - \frac{1}{p}}{1 - \left(\frac{\Delta c^*}{p}\right)^{\frac{1}{p}}},$$

where $c^* = \lambda c$ and γ is Euler's constant.

Note that this is close to being best possible, cf. (5).

Proof of Theorem 3.1. The theorem is a generalization of [12, Theorem 2] which corresponds

to the case $\Delta = 1$. As the proof is a modification of Joshi's argument, we give the necessary changes.

Fix some (small) $\epsilon > 0$. It suffices to show that for every (large) x there exists a prime $q \leq x$, $q \equiv c \pmod{d}$, such that

$$(7) \quad \begin{aligned} \log L(1, \chi) &\geq \log \log \log x + \gamma + \sum_{p|d} \log \left(1 - \frac{1}{p}\right) \\ &\quad - \sum_{p|d} \log \left(1 - \left(\frac{\Delta c^*}{p}\right) \frac{1}{p}\right) + \log(1 - 2\epsilon) + o(1). \end{aligned}$$

We prove (7) by constructing a set $\Sigma = \Sigma(x)$ of primes $q \leq x$, $q \equiv c \pmod{d}$, with $S = |\Sigma|$ and showing

$$(8) \quad \begin{aligned} \sum_{q \in \Sigma} \log L(1, \chi) &\geq S \left(\log \log \log x + \gamma + \sum_{p|d} \log \left(1 - \frac{1}{p}\right) \right. \\ &\quad \left. - \sum_{p|d} \log \left(1 - \left(\frac{\Delta c^*}{p}\right) \frac{1}{p}\right) + \log(1 - 2\epsilon) \right) + o(S). \end{aligned}$$

Put

$$y = (\log x)^{1-2\epsilon}$$

and let p_1, \dots, p_m be the primes not greater than y and not dividing d . Define r as in [12, p. 64], and let $k = dp_1 \cdots p_{r-1} p_{r+1} \cdots p_m$. For each $i \neq r$, let g_i (respectively h_i) be a quadratic residue (respectively non-residue) modulo p_i . Let $l \leq k$ be the unique positive integer satisfying $l \equiv c \pmod{d}$ and

$$l \equiv \begin{cases} g_i \pmod{p_i} & \text{for } \left(\frac{\lambda \Delta}{p_i}\right) = 1, i \neq r \\ h_i \pmod{p_i} & \text{for } \left(\frac{\lambda \Delta}{p_i}\right) = -1, i \neq r. \end{cases}$$

Define

$$\Sigma = \{q \text{ prime} \mid \sqrt{x} \leq q \leq x, q \equiv l \pmod{k}\}.$$

Then every $q \in \Sigma$ satisfies $q \equiv c \pmod{d}$ and $\chi(p_i) = 1$ for $i \neq r$ since

$$\chi(p_i) = \left(\frac{\Delta q^*}{p_i}\right) = \left(\frac{\lambda \Delta}{p_i}\right) \left(\frac{q}{p_i}\right) = \left(\frac{\lambda \Delta}{p_i}\right) \left(\frac{l}{p_i}\right) = 1.$$

So far, the only difference compared with Joshi's proof is the definition of l and χ (in [12], χ is the character $\left(\frac{\cdot}{q}\right) = \left(\frac{q^*}{\cdot}\right)$ corresponding to $\Delta = 1$). The different definition of l plays no role other than guaranteeing that we still have $\chi(p_i) = 1$, cf. [12, p. 65]. Hence, as in [12, (24)], we get

$$\begin{aligned} \sum_{q \in \Sigma} \log L(1, \chi) &\geq S \left(\log \log \log x + \gamma + \sum_{p|d} \log \left(1 - \frac{1}{p}\right) \right. \\ &\quad \left. - \sum_{p|d} \log \left(1 - \left(\frac{\Delta c^*}{p}\right) \frac{1}{p}\right) + \log(1 - 2\epsilon) \right) + R + o(S) \end{aligned}$$

where

$$R = \sum_{q \in \Sigma} \sum_{p > y} \frac{\chi(p)}{p}.$$

We now show $R = o(S)$ and hence (8) by splitting the summation over p into five intervals I_1, \dots, I_5 and thus writing $R = R_1 + \dots + R_5$ with

$$R_i = \sum_{q \in \Sigma} \sum_{p \in I_i} \frac{\chi(p)}{p}.$$

The estimation of R_1 and R_2 is practically the same as in Joshi's paper, only one has to replace $\left(\frac{\Delta}{p}\right)$ by $\left(\frac{\lambda\Delta}{p}\right)$ in [12, (27)] and the equation below that, which makes no difference since the sign of that factor plays no role anyway. The estimation of R_3 is exactly the same since it relies on the majorization

$$\left| \sum_{p \in I_3} \frac{\chi(p)}{p} \right| \leq \sum_{p \in I_3} \frac{1}{p}.$$

The estimation of R_4 requires some more care since it relies on the large sieve as stated in [12, Lemma 1] which works only for prime moduli. Put $\beta = 2 + \epsilon^{-1}$ and subdivide I_4 into intervals J_t each containing Z_t primes as in [12, p. 70]. Then [12, (30)] remains valid, i.e.

$$(9) \quad \sum_{q \in \Sigma} \sum_{p \in J_t} \frac{\chi(p)}{p} - \frac{1}{t} \sum_{q \in \Sigma} \sum_{p \in J_t} \chi(p) = O\left(\frac{S}{(\log x)^{2\beta}}\right).$$

Let J_t^+ and J_t^- denote the sets of primes in J_t with $\left(\frac{\Delta}{p}\right) = 1$ and $\left(\frac{\Delta}{p}\right) = -1$, respectively. Then $Z_t = Z_t^+ + Z_t^-$ where Z_t^+ and Z_t^- are defined analogously. Also, let $Z_t(a, q)$ be the number of p in J_t which are congruent to a modulo q , and similarly write $Z_t(a, q) = Z_t^+(a, q) + Z_t^-(a, q)$. Then a computation using the large sieve, cf. [12, p. 71], shows

$$\begin{aligned} \left| \frac{1}{t} \sum_{q \in \Sigma} \sum_{p \in J_t^+} \left(\frac{p}{q}\right) \right|^2 &= \frac{1}{t^2} \left| \sum_{q \in \Sigma} \sum_{j=1}^{q-1} \left(\frac{j}{q}\right) \left(Z_t^+(j, q) - \frac{Z_t^+}{q} \right) \right|^2 \\ &\leq \frac{S^2}{(\log x)^{4\beta}}, \end{aligned}$$

and similarly with the summation over $p \in J_t^-$. Since $\chi(p) = \left(\frac{\Delta}{p}\right)\left(\frac{p}{q}\right)$, we now get

$$\begin{aligned}
 \left| \frac{1}{t} \sum_{q \in \Sigma} \sum_{p \in J_t} \chi(p) \right| &\leq \left| \frac{1}{t} \sum_{q \in \Sigma} \sum_{p \in J_t^+} \chi(p) \right| + \left| \frac{1}{t} \sum_{q \in \Sigma} \sum_{p \in J_t^-} \chi(p) \right| \\
 &= \left| \frac{1}{t} \sum_{q \in \Sigma} \sum_{p \in J_t^+} \left(\frac{p}{q}\right) \right| + \left| \frac{1}{t} \sum_{q \in \Sigma} \sum_{p \in J_t^-} \left(\frac{p}{q}\right) \right| \\
 &\leq 2 \cdot \frac{S}{(\log x)^{2\beta}} \\
 (10) \qquad \qquad \qquad &= O\left(\frac{S}{(\log x)^{2\beta}}\right).
 \end{aligned}$$

From (9) and (10) follows

$$\left| \sum_{q \in \Sigma} \sum_{p \in J_t} \frac{\chi(p)}{p} \right| = O\left(\frac{S}{(\log x)^{2\beta}}\right),$$

and thus

$$R_4 = \sum_{q \in \Sigma} \sum_{p \in I_4} \frac{\chi(p)}{p} = o(S).$$

Finally, the estimation of R_5 can be carried out as in [12, p. 72] by writing

$$\sum_{v < p \leq w} \frac{\chi(p)}{p} = \sum_{j=1}^{|\Delta q|} \chi(j) \sum_{\substack{v < p \leq w \\ p \equiv j \pmod{|\Delta q|}}} \frac{1}{p}$$

and using [12, Lemma 3]. □

Proof of Theorem 1.1. If Δ is a negative fundamental discriminant, then let $c = 1$ and d be divisible by Δ and 8. Hence Theorem 3.1 implies that for $\chi = \chi_D$, $D = \Delta q$, we have

$$\sup_{\substack{q \rightarrow \infty \\ q \equiv c \pmod{d}}} L(1, \chi) = \infty.$$

Applying this to (4) yields the first statement. If Δ is a positive fundamental discriminant or 1, let $c = -1$ and d be divisible by Δ and 8. Applying Theorem 3.1 with $D = -\Delta q$ to (4) implies the second statement. □

5. OUTLOOK

We focused on large values of $C(\Delta q)$. Likewise, one might ask about small values. Is it true for example that $\liminf C(\Delta q) = 0$ as q runs over the primes congruent to 1 modulo 4? Is $C(\Delta q)$ constant on average? Likewise one can consider our table and wonder whether the values in a column are constant on average. If so, will this constant be zero or not? Perhaps variations of the techniques in [8] can be used to study this. A further open problem is to determine whether $B_f(x)$ is asymptotically better approximated by $C(D) \int_2^x dt \log^{-1/2} t$ or $C(D)x \log^{-1/2} x$. Finally, it

might be of interest to recover $C(D)$ for any discriminant $D < 0$ using sieve methods. For example, the Landau-Ramanujan constant C was verified (see [10]) using the half-dimensional sieve.

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REFERENCES

- [1] P. Bateman, S. Chowla and P. Erdős, Remarks on the size of $L(1, \chi)$, *Publ. Math. Debrecen* **1** (1950), 165–182.
- [2] P. Bernays, Über die Darstellung von positiven, ganzen Zahlen durch die primitiven, binären quadratischen Formen einer nicht-quadratischen Diskriminante, Dissertation, Göttingen, 1912, available at <http://www.math.uni-bielefeld.de/~rehmann/DML/>
- [3] S. Chowla, On the class-number of the corpus $P(\sqrt{-k})$, *Proc. Nat. Acad. Sci. India* **13** (1947), 197–200.
- [4] R. Cook, A note on character sums, *J. Number Theory* **11** (1979), 505–515.
- [5] D. Cox, *Primes of the Form $x^2 + ny^2$* , John Wiley & Sons, Inc, New York, 1989.
- [6] S. Finch, G. Martin and P. Sebah, Roots of unity and nullity modulo n , *Proc. Amer. Math. Soc.* **138** (2010), 2729–2743.
- [7] O. Fomenko, Distribution of values of Fourier coefficients of modular forms of weight 1, *J. Math. Sci. (New York)* **89** (1998), 1050–1071.
- [8] A. Granville and K. Soundararajan, The distribution of values of $L(1, \chi_d)$, *Geom. Funct. Anal.* **13** (2003), 992–1028.
- [9] G. Hardy, *Ramanujan*, Chelsea, New York, 1959.
- [10] H. Iwaniec, The half dimensional sieve, *Acta Arith.* **29** (1976), no. 1, 69–95.
- [11] R. James, The distribution of integers represented by quadratic forms, *Amer. J. Math.* **60** (1938), 737–744.
- [12] P. Joshi, The size of $L(1, \chi)$ for real nonprincipal residue characters χ with prime modulus, *J. Number Theory* **2** (1970), 58–73.
- [13] P. Kaplan and K. Williams, The genera representing a positive integer, *Acta Arith.* **102** (2002), 353–361.
- [14] E. Landau, Über die Einteilung der positiven ganzen Zahlen in vier Klassen nach der Mindestzahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate, *Arch. der Math. und Phys. (3)* **13** (1908), 305–312.
- [15] J. Littlewood, On the class-number of the corpus $P(\sqrt{-k})$, *Proc. London Math. Soc. (2)* **27** (1928), 358–372.
- [16] P. Moree and R. Osburn, Two-dimensional lattices with few distances, *Enseignement Math.* **52** (2006), 361–380.
- [17] D. Shanks, The second order term in the asymptotic expansion of $B(x)$, *Math. Comp.* **18** (1964), 75–86.
- [18] D. Shanks and L. Schmid, Variations on a theorem of Landau. I, *Math. Comp.* **20** (1966), 551–569.
- [19] Z. Sun and K. Williams, On the number of representations of n by $ax^2 + bxy + cy^2$, *Acta Arith.* **122** (2006), 101–171.
- [20] K. Williams, Note on integers representable by binary quadratic forms, *Canad. Math. Bull.* **18** (1975), 123–125.

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