QUADRATIC FORMS AND FOUR PARTITION FUNCTIONS MODULO 3

JEREMY LOVEJOY AND ROBERT OSBURN

Abstract. Recently, Andrews, Hirschhorn and Sellers have proven congruences modulo 3 for four types of partitions using elementary series manipulations. In this paper, we generalize their congruences using arithmetic properties of certain quadratic forms.

1. Introduction

A partition of a non-negative integer n is a non-increasing sequence whose sum is n. An overpartition of n is a partition of n where we may overline the first occurrence of a part. Let $\overline{p}(n)$ denote the number of overpartitions of n, $\overline{p_o}(n)$ the number of overpartitions of n into odd parts, $ped(n)$ the number of partitions of n without repeated even parts and $pod(n)$ the number of partitions of n without repeated odd parts. The generating functions for these partitions are

$$\sum_{n\geq 0}\overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}, \tag{1.1}$$

$$\sum_{n\geq 0}\overline{p_o}(n)q^n = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.2}$$

$$\sum_{n\geq 0}ped(n)q^n = \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.3}$$

$$\sum_{n\geq 0}pod(n)q^n = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}, \tag{1.4}$$

where as usual

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

The infinite products in (1.1)–(1.4) are essentially the four different ways one can specialize the product $(-aq;q)_\infty/(bq;q)_\infty$ to obtain a modular form whose level is relatively prime to 3.

A series of four recent papers examined congruence properties for these partition functions modulo 3 [1, 5, 6, 7]. Among the main theorems in these papers are the following congruences (see Theorem 1.3 in [6], Corollary 3.3 and Theorem 3.5 in [1], Theorem 1.1 in [5] and Theorem 3.2 in [7], respectively). For all $n \geq 0$ and $\alpha \geq 0$ we have

$$\overline{p_o}(3^{2\alpha}(An+B)) \equiv 0 \pmod 3, \tag{1.5}$$

where $An + B = 9n + 6$ or $27n + 9$,

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$$ped\Big(3^{2\alpha+3}n + \frac{17\cdot 3^{2\alpha+2}-1}{8}\Big) \equiv ped\Big(3^{2\alpha+2}n + \frac{19\cdot 3^{2\alpha+1}-1}{8}\Big) \equiv 0 \pmod 3, \tag{1.6}$$

$$\overline{p}(3^{2\alpha}(27n+18)) \equiv 0 \pmod 3 \tag{1.7}$$

and

$$pod\left(3^{2\alpha+3} + \frac{23\cdot 3^{2\alpha+2}+1}{8}\right) \equiv 0 \pmod 3. \tag{1.8}$$

We note that congruences modulo 3 for $\overline{p}(n)$, $\overline{p}_o(n)$ and $ped(n)$ are typically valid modulo 6 or 12. The powers of 2 enter trivially (or nearly so), however, so we do not mention them here.

The congruences in (1.5)–(1.8) are proven in [1, 5, 6, 7] using elementary series manipulations. If we allow ourselves some elementary number theory, we find that much more is true.

With our first result we exhibit formulas for $\overline{p}_o(3n)$ and $ped(3n+1)$ modulo 3 for all $n \geq 0$. These formulas depend on the factorization of n, which we write as

$$n = 2^a 3^b \prod_{i=1}^{r} p_i^{v_i} \prod_{j=1}^{s} q_j^{w_j}, \tag{1.9}$$

where $p_i \equiv 1$, 5, 7 or 11 $\pmod{24}$ and $q_j \equiv 13$, 17, 19 or 23 $\pmod{24}$. Further, let t denote the number of prime factors of n (counting multiplicity) that are congruent to 5 or 11 $\pmod{24}$. Let $R(n, Q)$ denote the number of representations of n by the quadratic form Q.

Theorem 1.1. *For all $n \geq 0$ we have*

$$\overline{p}_o(3n) \equiv f(n) R(n, x^2+6y^2) \pmod 3$$

and

$$ped(3n+1) \equiv (-1)^{n+1} R(8n+3, 2x^2+3y^2) \pmod 3,$$

where $f(n)$ is defined by

$$f(n) = \begin{cases} -1, & n \equiv 1, 6, 9, 10 \pmod{12}, \\ 1, & \textit{otherwise.} \end{cases}$$

Moreover, we have

$$\overline{p}_o(3n) \equiv f(n)(1+(-1)^{a+b+t}) \prod_{i=1}^{r}(1+v_i) \prod_{j=1}^{s}\left(\frac{1+(-1)^{w_j}}{2}\right) \pmod 3 \tag{1.10}$$

and

$$(-1)^n ped(3n+1) \equiv \overline{p}_o(48n+18) \pmod 3. \tag{1.11}$$

There are many ways to deduce congruences from Theorem 1.1. For example, calculating the possible residues of x^2+6y^2 modulo 9 we see that

$$R(3n+2, x^2+6y^2) = R(9n+3, x^2+6y^2) = 0,$$

and then (1.10) implies that $\overline{p}_o(27n) \equiv \overline{p}_o(3n) \pmod 3$. This gives (1.5). The congruences in (1.6) follow from those in (1.5) after replacing $48n+18$ by $3^{2\alpha}(48(3n+2)+18)$ and $3^{2\alpha}(48(9n+6)+18)$ in (1.11). We record two more corollaries, which also follow readily from Theorem 1.1.

Corollary 1.2. *For all $n \geq 0$ and $\alpha \geq 0$ we have*

$$\overline{p_o}(2^{2\alpha}(An+B)) \equiv 0 \pmod{3},$$

where $An+B = 24n+9$ or $24n+15$.

Corollary 1.3. *If $\ell \equiv 1, 5, 7$ or $11 \pmod{24}$ is prime, then for all n with $\ell \nmid n$ we have*

$$\overline{p}_o(3\ell^2 n) \equiv 0 \pmod{3}. \tag{1.12}$$

For the functions $\overline{p}(3n)$ and $pod(3n+2)$ we have relations not to binary quadratic forms but to $r_5(n)$, the number of representations of n as the sum of five squares. Our second result is the following.

Theorem 1.4. *For all $n \geq 0$ we have*

$$\overline{p}(3n) \equiv (-1)^n r_5(n) \pmod{3}$$

and

$$pod(3n+2) \equiv (-1)^n r_5(8n+5) \pmod{3}.$$

Moreover, for all odd primes ℓ and $n \geq 0$, we have

$$\overline{p}(3\ell^2 n) \equiv \left(\ell - \ell\left(\frac{n}{\ell}\right) + 1\right)\overline{p}(3n) - \ell\overline{p}(3n/\ell^2) \pmod{3} \tag{1.13}$$

and

$$(-1)^{n+1} pod(3n+2) \equiv \overline{p}(24n+15) \pmod{3}, \tag{1.14}$$

where $\left(\frac{\bullet}{\ell}\right)$ denotes the Legendre symbol.

Here we have taken $\overline{p}(3n/\ell^2)$ to be 0 unless $\ell^2 \mid 3n$. Again there are many ways to deduce congruences. For example, (1.7) follows readily upon combining (1.13) in the case $\ell = 3$ with the fact that

$$r_5(9n+6) \equiv 0 \pmod{3},$$

which is a consequence of the fact that $R(9n+6, x^2+y^2+3z^2) = 0$. One can check that (1.8) follows similarly. For another example, we may apply (1.13) with n replaced by $n\ell$ for $\ell \equiv 2 \pmod{3}$ to obtain

Corollary 1.5. *If $\ell \equiv 2 \pmod{3}$ is prime and $\ell \nmid n$, then*

$$\overline{p}(3\ell^3 n) \equiv 0 \pmod{3}.$$

2. Proofs of Theorems 1.1 and 1.4

Proof of Theorem 1.1. On page 364 of [6] we find the identity

$$\sum_{n\geq 0} \overline{p}_o(3n) q^n = \frac{D(q^3)D(q^6)}{D(q)^2},$$

where

$$D(q) := \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}.$$

Reducing modulo 3, this implies that

$$\begin{aligned}\sum_{n\geq 0}\overline{p_o}(3n)q^n &\equiv \sum_{x,y\in\mathbb{Z}}(-1)^{x+y}q^{x^2+6y^2} \pmod 3\\ &\equiv \sum_{n\geq 0} f(n)R(n,x^2+6y^2)q^n \pmod 3.\end{aligned}$$

Now it is known (see Corollary 4.2 of [3], for example) that if n has the factorization in (1.9), then

$$R(n,x^2+6y^2) = (1+(-1)^{a+b+t})\prod_{i=1}^{r}(1+v_i)\prod_{j=1}^{s}\left(\frac{1+(-1)^{w_j}}{2}\right). \tag{2.1}$$

This gives (1.10). Next, from [1] we find the identity

$$\sum_{n\geq 0} ped(3n+1)q^n = \frac{D(q^3)\psi(-q^3)}{D(q)^2},$$

where

$$\psi(q) := \sum_{n\geq 0} q^{n(n+1)/2}.$$

Reducing modulo 3, replacing q by $-q^8$ and multiplying by q^3 gives

$$\sum_{n\geq 0}(-1)^{n+1}ped(3n+1)q^{8n+3} \equiv \sum_{n\geq 0} R(8n+3,2x^2+3y^2)q^{8n+3} \pmod 3.$$

It is known (see Corollary 4.3 of [3], for example) that if n has the factorization given in (1.9), then

$$R(n,2x^2+3y^2) = (1-(-1)^{a+b+t})\prod_{i=1}^{r}(1+v_i)\prod_{j=1}^{s}\left(\frac{1+(-1)^{w_j}}{2}\right).$$

Comparing with (2.1) finishes the proof of (1.11). □

Proof of Theorem 1.4. On page 3 of [5] we find the identity

$$\sum_{n\geq 0}\overline{p}(3n)q^n \equiv \frac{D(q^3)^2}{D(q)} \pmod 3.$$

Reducing modulo 3 and replacing q by $-q$ yields

$$\sum_{n\geq 0}(-1)^n\overline{p}(3n)q^n \equiv \sum_{n\geq 0} r_5(n)q^n \pmod 3.$$

It is known (see Lemma 1 in [4], for example) that for any odd prime ℓ we have

$$r_5(\ell^2 n) = \left(\ell^3 - \ell\left(\frac{n}{\ell}\right)+1\right)r_5(n) - \ell^3 r_5(n/\ell^2).$$

Here $r_5(n/\ell^2) = 0$ unless $\ell^2 \mid n$. Replacing $r_5(n)$ by $(-1)^n\overline{p}(3n)$ throughout gives (1.13). Now equation (1) of [7] reads

$$\sum_{n\geq 0}(-1)^n pod(3n+2)q^n = \frac{\psi(q^3)^3}{\psi(q)^4}.$$

Reducing modulo 3 we have

$$\begin{aligned}\sum_{n\geq 0}(-1)^n pod(3n+2)q^n &\equiv \psi(q)^5 \pmod 3\\ &\equiv \sum_{n\geq 0} r_5(8n+5)q^n \pmod 3\\ &\equiv -\sum_{n\geq 0}\overline{p}(24n+15)q^n \pmod 3,\end{aligned}$$

where the second congruence follows from Theorem 1.1 in [2]. This implies (1.14) and thus the proof of Theorem 1.4 is complete.

□

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CNRS, LIAFA, Université Denis Diderot - Paris 7, Case 7014, 75205 Paris Cedex 13, FRANCE

School of Mathematical Sciences, University College Dublin, Belfield, Dublin 4, Ireland
E-mail address: lovejoy@liafa.jussieu.fr
E-mail address: robert.osburn@ucd.ie