## Fronts \& Frontogenesis

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In a landmark paper, Sawyer (1956) stated that
"although the Norwegian system of frontal analysis has been generally accepted by weather forecasters since the 1920's, no satisfactory explanation has been given for the up-gliding motion of the warm air to which is attributed the characteristic frontal cloud and rain. Simple dynamical theory shows that a sloping discontinuity between two air masses with different densities and velocities can exist without vertical movement of either air mass ...".

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Sawyer goes on to suggest that
". .. a front should be considered not so much as a stable area of strong temperature contrast between two air masses, but as an area into which active confluence of air currents of different temperature is taking place."

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Therefore, clearly defined fronts are likely to be found only where active frontogenesis is in progress; i.e., in an area where the horizontal air movements are such as to intensify the horizontal temperature gradients.

These ideas are supported by observations.

## Kinematics of Frontogenesis

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Examples of two basic horizontal flow configurations which can lead to frontogenesis are shown below.


The intensification of horizontal temperature by horizontal shear, and pure horizontal deformation.

A parallel shear flow and a pure deformation field can intensify temperature gradients provided the isotherms are suitably oriented.

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To understand the way in which motion fields in general lead to frontogenesis and, indeed, to quantify the rate of frontogenesis, we need to study the relative motion near a point $P$ in a fluid, as indicated in the following figure.


Let $\mathbf{P}$ be at $(x, y)$ and $\mathbf{Q}$ at $(x+\delta x, y+\delta y)$. Let the velocity at $\mathbf{P}$ be $\left(u_{0}, v_{0}\right)$ and that at $\mathbf{Q}$ be $\left(u_{0}+\delta u, v_{0}+\delta v\right)$.

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The relative motion between the flow at P and at the neighbouring point Q is

$$
\delta u=u-u_{0} \approx \frac{\partial u}{\partial x} \delta x+\frac{\partial u}{\partial y} \delta y \quad \delta v=v-v_{0} \approx \frac{\partial v}{\partial x} \delta x+\frac{\partial v}{\partial y} \delta y
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$$

In matrix form, this is

$$
\binom{\delta u}{\delta v}=\left[\begin{array}{l}
\frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x}
\end{array} \frac{\partial v}{\partial y}\right]\binom{\delta x}{\delta y}=\left[\begin{array}{ll}
u_{x} & u_{y} \\
v_{x} & v_{y}
\end{array}\right]\binom{\delta x}{\delta y}=\mathrm{M}\binom{\delta x}{\delta y}
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$$

Any matrix can be written as a sum of a symmetric matrix and an antisymmetric matrix:

$$
M=\frac{1}{2}\left(M+M^{\top}\right)+\frac{1}{2}\left(M-M^{\top}\right)
$$

We introduce a pair of matrices, S:

$$
\mathbf{S}=\frac{1}{2}\left(\mathbf{M}+\mathbf{M}^{\boldsymbol{\top}}\right)=\left[\begin{array}{ll}
\frac{1}{2}\left(u_{x}+u_{x}\right) & \frac{1}{2}\left(u_{y}+v_{x}\right) \\
\frac{1}{2}\left(v_{x}+u_{y}\right) & \frac{1}{2}\left(v_{y}+v_{y}\right)
\end{array}\right]
$$

and $A$ :

$$
\mathbf{A}=\frac{1}{2}\left(\mathbf{M}-\mathbf{M}^{\boldsymbol{\top}}\right)=\left[\begin{array}{ll}
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Such a decomposition is standard in developing the equations for viscous fluid motion (see e.g. Batchelor, 1970, 2.3).

It can be shown that $S$ and $A$ are second order tensors. $S$ is symmetric $\left(\mathrm{S}_{j i}=\mathrm{S}_{i j}\right)$ and A antisymmetric $\left(\mathrm{A}_{j i}=-\mathrm{A}_{i j}\right)$.

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Note that $A$ has only one independent non-zero component, equal to half the vertical component of vorticity, $\frac{1}{2} \zeta$.

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We can write

$$
\begin{aligned}
& \delta u=\left(S_{11} \delta x+S_{12} \delta y\right)+\left(A_{11} \delta x+A_{12} \delta y\right) \\
& \delta v=\left(S_{21} \delta x+S_{22} \delta y\right)+\left(A_{21} \delta x+A_{22} \delta y\right)
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Using the fact that $A_{11}$ and $A_{22}$ are zero, we have

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\begin{aligned}
& \delta u=S_{11} \delta x+\left(S_{12}+A_{12}\right) \delta y \\
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$$

We now locate the origin of coordinates at the point P , so that $(\delta x, \delta y)$ become simply $(x, y)$.
Also, $A_{21}=\frac{1}{2}\left(v_{x}-u_{y}\right)=\frac{1}{2} \zeta$ and $A_{12}=-A_{21}=-\frac{1}{2} \zeta$.

Now, in preference to the four derivatives $u_{x}, u_{y}, v_{x}, v_{y}$, we define the equivalent four combinations of these derivatives:

$$
\begin{aligned}
D & =u_{x}+v_{y}, \\
E & \text { the divergence (formerly } \delta \text { ) } \\
F & =u_{x}-v_{y}, u_{y}, \\
\zeta & \text { the stretching deformation } \\
\zeta=v_{x}-u_{y}, & \text { the vorticity }
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$$

Obviously, we can solve for $u_{x}, u_{y}, v_{x}$ and $v_{y}$ as functions of $D, E, F$ and $\zeta$ :

$$
u_{x}=\frac{1}{2}(D+E), \quad u_{y}=\frac{1}{2}(F-\zeta), \quad v_{x}=\frac{1}{2}(F+\zeta), \quad v_{y}=\frac{1}{2}(D-E)
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Note that $E$ is like $D$, but with a minus sign; $F$ is like $\zeta$, but with a plus sign.

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Note that $E$ is like $D$, but with a minus sign; $F$ is like $\zeta$, but with a plus sign.
$E$ is called the stretching deformation because the velocity components are differentiated in the direction of the component. In $F$, the shearing deformation, each velocity component is differentiated at right angles to its direction.

We can write the relative velocity as

$$
\binom{\delta u}{\delta v}=\left[\begin{array}{cc}
u_{x} & \frac{1}{2}\left(v_{x}+u_{y}\right)-\frac{1}{2} \zeta \\
\frac{1}{2}\left(v_{x}+u_{y}\right)+\frac{1}{2} \zeta & v_{y}
\end{array}\right]\binom{x}{y}
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$$

This equation may now be written in the form:

$$
\binom{\delta u}{\delta v}=\frac{1}{2}\left[\left(\begin{array}{cc}
D & 0 \\
0 & D
\end{array}\right)+\left(\begin{array}{cc}
E & 0 \\
0 & -E
\end{array}\right)+\left(\begin{array}{cc}
0 & F \\
F & 0
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\end{array}\right)\right]\binom{x}{y}
$$

In component form, this is

$$
\begin{aligned}
& u=u_{0}+\frac{1}{2}(D x+E x+F y-\zeta y) \\
& v=v_{0}+\frac{1}{2}(D y-E y+F x+\zeta x)
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where $\delta u=u-u_{0}, \delta v=v-v_{0}$, and $\left(u_{0}, v_{0}\right)$ is the translation velocity at the point $P$.

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\end{aligned}
$$

where $\delta u=u-u_{0}, \delta v=v-v_{0}$, and $\left(u_{0}, v_{0}\right)$ is the translation velocity at the point P .
Henceforth, we choose our frame of reference so that $u_{0}=v_{0}=0$. That is, the frame moves with the point $\mathbf{P}$.


Schematic diagram of the components of flow in the neighbourhood of a point: (a) pure divergence/convergence; (b) pure rotation; (c) pure stretching deformation; and (d) pure shearing deformation.

## Decomposition of Relative Motion

Clearly, the relative motion near the point $P$ can be decomposed into four basic components, as follows.

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(I) Pure divergence (only $D$ nonzero). Then $u=\frac{1}{2} D x, v=$ $\frac{1}{2} D y$ or, in vector notation, $\mathbf{u}=\frac{1}{2} D(r \cos \theta, r \sin \theta)=D \mathbf{r}, \mathbf{r}$ being the position vector from $P$. Thus the motion is purely radial and is to or from the point $P$ according to the sign of $D$.

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(II) Pure rotation (only $\zeta$ nonzero). Then $u=-\frac{1}{2} \zeta y, v=\frac{1}{2} \zeta x$, whereupon $u=\frac{1}{2}(-r \sin \theta, r \cos \theta)=\frac{1}{2} \zeta r \hat{\theta}$, where $\hat{\theta}$ is the unit normal vector to r . Clearly such motion corresponds with solid body rotation with angular velocity $\frac{1}{2} \zeta$.


Schematic diagram of the components of flow in the neighbourhood of a point: (a) pure divergence/convergence; (b) pure rotation; (c) pure stretching deformation; and (d) pure shearing deformation.
(III) Pure stretching deformation (only $E$ nonzero). The velocity components are given by

$$
u=\frac{1}{2} E x, \quad v=-\frac{1}{2} E y
$$

On a streamline, $d y / d x=v / u=-y / x$, or $x d y+y d x=d(x y)=$ 0 . Hence the streamlines are rectangular hyperbolae $x y=$ constant. In the figure, the indicated flow directions are for $E>0$. For $E<0$, the directions are reversed.
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(IV) Pure shearing deformation (only $F$ nonzero). The velocity components are given by

$$
u=\frac{1}{2} F y, \quad v=\frac{1}{2} F x
$$

The streamlines are given now by $d y / d x=x / y$, or $d\left(y^{2}-\right.$ $\left.x^{2}\right)=0$, so that $y^{2}-x^{2}=$ constant. Thus the streamlines are again rectangular hyperbolae, but with their axes of dilatation and contraction at $45^{\circ}$ to the coordinate axes. The flow directions indicated are for $F>0$.


Schematic diagram of the components of flow in the neighbourhood of a point: (a) pure divergence/convergence; (b) pure rotation; (c) pure stretching deformation; and (d) pure shearing deformation.

Total deformation. We assume that $\zeta=D=0$ and that $E$ and $F$ are nonzero). Then

$$
\begin{aligned}
& \delta u=\frac{1}{2}(+E x+F y) \\
& \delta v=\frac{1}{2}(-E y+F x)
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We can show, by rotating the axes $(x, y)$ to $\left(x^{\prime}, y^{\prime}\right)$, that we can choose the rotation angle $\phi$ so that the two deformation fields together reduce to a single field with the axis of dilatation at angle $\phi$ to the $x$-axis.
[For details, see Roger Smith's notes, pp. 176-177].

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[For details, see Roger Smith's notes, pp. 176-177].
In other words, the stretching and shearing deformation fields may be combined to give a single total deformation field. The strength of this field is given by

$$
E^{\prime}=\left(E^{2}+F^{2}\right)^{1 / 2}
$$

and the axis of dilatation is inclined at an angle $\phi$ to the $x$-axis given by

$$
\tan 2 \phi=F / E
$$

The total deformation field is illustrated below.


## The Frontogenesis Function

The frontogenetic or frontolytic tendency in a flow can be measured by the quantity $d\left|\nabla_{h} \theta\right| / d t$, which is called the frontogenesis function.

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Differentiating with respect to $x$ and $y$ in turn gives

$$
\begin{aligned}
& \frac{d}{d t}\left(\frac{\partial \theta}{\partial x}\right)+\frac{\partial u}{\partial x} \frac{\partial \theta}{\partial x}+\frac{\partial v}{\partial x} \frac{\partial \theta}{\partial y}+\frac{\partial w}{\partial x} \frac{\partial \theta}{\partial z}=\frac{\partial \dot{q}}{\partial x} \\
& \frac{d}{d t}\left(\frac{\partial \theta}{\partial y}\right)+\frac{\partial u}{\partial y} \frac{\partial \theta}{\partial x}+\frac{\partial v}{\partial y} \frac{\partial \theta}{\partial y}+\frac{\partial w}{\partial y} \frac{\partial \theta}{\partial z}=\frac{\partial \dot{q}}{\partial y}
\end{aligned}
$$

But we have

$$
\frac{d}{d t}|\nabla \theta|^{2}=2 \nabla \theta \cdot \frac{d}{d t} \nabla \theta=2\left(\frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}\right) \cdot\left[\frac{d}{d t}\left(\frac{\partial \theta}{\partial x}\right), \frac{d}{d t}\left(\frac{\partial \theta}{\partial y}\right)\right]
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Substituting from above we get

$$
\frac{d}{d t}|\nabla \theta|^{2}=2\left[\left(\theta_{x} \dot{q}_{x}+\theta_{y} \dot{q}_{y}\right)-\left(\theta_{x} w_{x}+\theta_{y} w_{y}\right) \theta_{z}-\left(u_{x} \theta_{x}^{2}+v_{y} \theta_{y}^{2}\right)-\left(v_{x}+u_{y}\right) \theta_{x} \theta_{y}\right]
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We now recall the formulae:

$$
u_{x}=\frac{1}{2}(D+E), \quad u_{y}=\frac{1}{2}(F-\zeta), \quad v_{x}=\frac{1}{2}(F+\zeta), \quad v_{y}=\frac{1}{2}(D-E)
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Note that the vorticity $\zeta$ does not appear in this equation.

There are four separate effects contributing to frontogenesis. Let us write

$$
\frac{d}{d t}|\nabla \theta|=T_{1}+T_{2}+T_{3}+T_{4}
$$

where

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& T_{1}=\left(\theta_{x} \dot{q}_{x}+\theta_{y} \dot{q}_{y}\right) /|\nabla \theta| \\
& T_{2}=-\left(\theta_{x} w_{x}+\theta_{y} w_{y}\right) \theta_{z} /|\nabla \theta| \\
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Defining $\hat{\mathbf{n}}$ to be the unit vector in the direction of $|\nabla \theta|$, we can write

$$
\begin{aligned}
& T_{1}=\hat{\mathbf{n}} \cdot \nabla \dot{q} \\
& T_{2}=-\theta_{z} \hat{\mathbf{n}} \cdot \nabla w \\
& T_{3}=-\frac{1}{2} D|\nabla \theta| \\
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T4: represents the frontogenetic effect of a (total) horizontal deformation field.
$\mathrm{T} 1: T_{1}=\hat{\mathbf{n}} \cdot \nabla \dot{q}$
The rate of frontogenesis due to a gradient of diabatic heating in the direction of the existing temperature gradient.


Cool
$\nabla_{\mathrm{h}} \dot{\mathrm{q}}$

## Heat

T2: $T_{2}=-\theta_{z} \hat{\mathbf{n}} \cdot \nabla w$
The conversion of vertical temperature gradient to horizontal gradient by a component of differential vertical motion (vertical shear) in the direction of the existing temperature gradient.

T3: $T_{3}=-\frac{1}{2} D|\nabla \theta|$
The rate of increase of horizontal temperature gradient due to horizontal convergence (i.e., negative divergence) in the presence of an existing gradient.


T4: $T_{4}=-\frac{1}{2}\left[E \theta_{x}^{2}+2 F \theta_{x} \theta_{y}-E \theta_{y}^{2}\right] /|\nabla \theta|$
The frontogenetic effect of a (total) horizontal deformation field.


Further insight into the term $T_{4}$ may be obtained by a rotation of axes to those of the deformation field. We can show that

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T_{4}=\frac{1}{2} E^{\prime}|\nabla \theta| \cos 2 \beta
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When the angle $\beta$ is between $45^{\circ}$ and $90^{\circ}$, deformation has a frontolytic effect, i.e., $T_{4}$ is negative.

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A case study by Ogura and Portis (1982) shows that $T_{2}, T_{3}$ and $T_{4}$ are all important in the immediate vicinity of the front, whereas this and other investigations suggest that horizontal deformation (including horizontal shear) plays a primary role on the synoptic scale.

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Clearly, on a large scale, term $T_{1}$ must be dominant. Why?

This Figure shows a mean-sea-level isobaric analysis for the Australian region with a cold front over south-eastern Australia sandwiched between two anticyclones.


This situation is frontogenetic with warm air advection in the hot northerlies ahead of the front and strong cold air advection in the maritime southwesterlies behind it.


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He found also that the effect is most clearly defined at the 700 mb level, at which the rate of contraction of fluid elements in the direction of the temperature gradient usually has a well-defined maximum near the front.

A graphic illustration of the way in which
 flow deformation acting on an advected passive scalar quantity produces locally large gradients of the scalar was given by Welander (1955).


