Synoptic Meteorology

[MAPH P312]

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Seminar Room
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Part 8

The Quasigeostrophic System

These lectures follow closely the text of Holton (Chapter 6).

The Quasi-Geostrophic Equations

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In that case, the three-dimensional flow is determined by the geopotential field.

The Primitive Equations

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The dynamical equations in pressure coordinates are

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k} \times \mathbf{V} + \nabla\Phi = 0$$

$$\frac{\partial\Phi}{\partial p} = -\frac{RT}{p}$$

$$\nabla\cdot\mathbf{V} + \frac{\partial\omega}{\partial p} = 0$$

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Here the total time derivative is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{V}\cdot\nabla)_p + \omega\frac{\partial}{\partial p}$$

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The primitive equations will now be simplified based on the assumption that the flow is close to geostrophic balance and the vertical velocity is much smaller than the horizontal.

The Momentum Equation

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We first partition the horizontal component of the wind into geostrophic and ageostrophic parts

$$\mathbf{V} = \mathbf{V_g} + \mathbf{V_a}$$

with the geostrophic wind defined by

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We take a constant "central" value f_0 of the Coriolis parameter here. This is consistent with the assumption that the horizontal scale L of the motion is small compared to the Earth's radius, $L \ll a$.

We note also that the geostrophic divergence vasnishes:

$$\delta_g = \nabla\cdot\mathbf{V_g} = \frac{\partial}{\partial x}\left(-\frac{1}{f_0}\frac{\partial\Phi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{1}{f_0}\frac{\partial\Phi}{\partial x}\right) = 0$$

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Moreover, if ζ_g is given, the Poisson equation

$$\nabla^2 \Phi = f_0 \zeta_g$$

may be solved for the geopotential. Then $\mathbf{V_g}$ follows immediately.

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Then the size of the advection relative to the Coriolis term is

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We can then replace the velocity by its geostrophic component, and ignore the vertical advection in the total time derivative:

$$\frac{d\mathbf{V}}{dt} \approx \left(\frac{d}{dt}\right)_g \mathbf{V_g} = \left(\frac{\partial}{\partial t} + \mathbf{V_g}\cdot\nabla\right)\mathbf{V_g}$$

Exercise: **Using the vector relationship $\mathbf{k} \cdot \nabla \times \mathbf{V} = \nabla \cdot \mathbf{V} \times \mathbf{k}$ and the definition $\mathbf{V_g} = (1/f_0)\mathbf{k} \times \nabla\Phi$, derive the above expression for the geostrophic vorticity.**

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Expanding in a Taylor series, we write the first two terms

$$f = f_0 + \beta y$$

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The ratio of the two terms is

$$\frac{\beta y}{f_0} \sim \frac{\cos\phi_0 L}{\sin\phi_0 a} \sim \frac{L}{a} \sim \mathbf{Ro} \ll 1$$

We now consider the Coriolis and pressure gradient terms

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$$\begin{aligned} f\mathbf{k} \times \mathbf{V} + \nabla\Phi &= (f_0 + \beta y)\mathbf{k} \times (\mathbf{V_g} + \mathbf{V_a}) - f_0\mathbf{k} \times \mathbf{V_g} \\ &\approx f_0\mathbf{k} \times \mathbf{V_a} + \beta y\,\mathbf{k} \times \mathbf{V_g} \end{aligned}$$

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The horizontal momentum equation may now be written

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All the terms here are $O(\mathbf{Ro})$ and neglected terms are $O(\mathbf{Ro}^2)$.

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We can use the geostrophic wind in the expression for horizontal advection.

Moreover, we separate the temperature field into a basic part varying only in the vertical and a part depending on all coordinates and time:

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We can replace T by T_0 and θ by θ_0 in evaluating the static stability:

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Note that S_0 depends only on pressure p.

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Then the thermodynamic equation becomes

$$\left(\frac{\partial}{\partial t} + \mathbf{V_g} \cdot \nabla\right)\left(\frac{\partial \Phi}{\partial p}\right) + \sigma\omega = -\frac{\kappa \dot{Q}}{p}$$

where $\kappa = R/c_p$ and σ is another measure of static stability:

$$\sigma \equiv \frac{R}{p} S_0 = -\frac{RT_0}{p}\frac{d\ln\theta_0}{dp}$$

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Although σ varies with height, *we will assume that it is a constant*. This simplifies the analysis.

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The complete system of Quasigeostrophic Equations is:

$$\left(\frac{\partial}{\partial t} + \mathbf{V_g} \cdot \nabla\right) \mathbf{V_g} + f_0 \mathbf{k} \times \mathbf{V_a} + \beta y \mathbf{k} \times \mathbf{V_g} = 0$$

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These 4 equations (6 scalar equations) form a complete system for the variables Φ, $\mathbf{V_g}$, $\mathbf{V_a}$ and ω (6 scalar variables).

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These 4 equations (6 scalar equations) form a complete system for the variables Φ, $\mathbf{V_g}$, $\mathbf{V_a}$ and ω (6 scalar variables).

However, they are not in a form convenient for prediction. For this purpose, we derive an equation for the geostrophic vorticity.