§5.5: Ekman Pumping

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$$D = \frac{\pi}{\gamma} = \pi\sqrt{\frac{2K}{f}} = \pi\sqrt{\frac{2\times 5}{10^{-4}}} = 993\,\mathrm{m} \approx 1\,\mathrm{km}$$

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Note that D depends on the values of f and K so the particular value 1 km is more an indication of the scale that a sharp quantitative estimate.

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The modified Ekman Layer is discussed on Holton (§5.3.6). We will not discuss it here.

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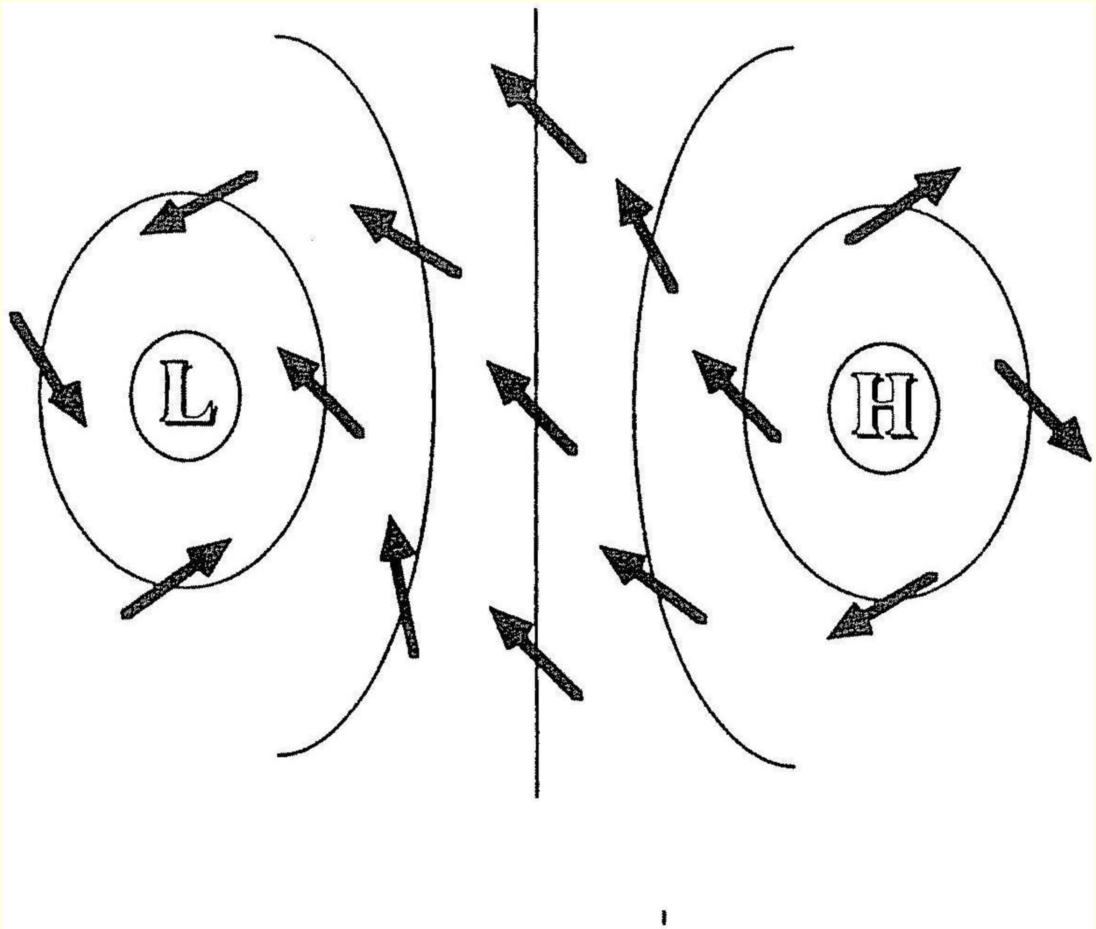

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We will now calculate the vertical velocity at the top of the Ekman layer.

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The result is thus

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Here we have used the fact that

$$e^{-\pi} \approx 0.0432 \ll 1$$

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- **Evaluate the integral analytically**
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Note that the analytical evaluation of the integral is straightforward.

For example, it can be done by means of integration by parts (twice), or by expressing the $\sin$-function in terms of complex exponentials.

Next, integrate the continuity equation through the PBL:

$$\int_0^D \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dz = \int_0^D \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz + [w(D) - w(0)] = 0$$

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Subsitituing this into the equation for $w(D)$ **gives**

$$w(D) = -\frac{\partial u_{\mathrm{g}}}{\partial y}\int_0^D e^{-\pi z/D}\sin(\pi z/D)\,dz = -\tfrac{1}{2}\left(\tfrac{D}{\pi}\right)\frac{\partial u_{\mathrm{g}}}{\partial y}$$

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Thus,

$$w(D) = -\tfrac{1}{2}\left(\tfrac{D}{\pi}\right)\tfrac{\partial u_{\mathrm{g}}}{\partial y} = \left(\tfrac{D}{2\pi}\right)\zeta_{\mathrm{g}}$$

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Note on Dines Mechanism to be added later.

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$$w(D) = \left(\frac{1}{2\pi}\right) \times 10^3 \times (5 \times 10^{-5}) = \frac{5 \times 10^{-2}}{2\pi} \approx 8\,\mathrm{mm\,s}^{-1} \sim 1\,\mathrm{cm\,s}^{-1}$$

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If it is sufficient to lift air to its LCL, then latent heat release allows stronger updrafts within the convective clouds.

Storms in Teacups

Standing waves in a tea cup, induced by the propeller rotation of an airoplane.

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The centrifugal force is given, as usual, by

$$\frac{V^2}{r} = \omega^2 r$$

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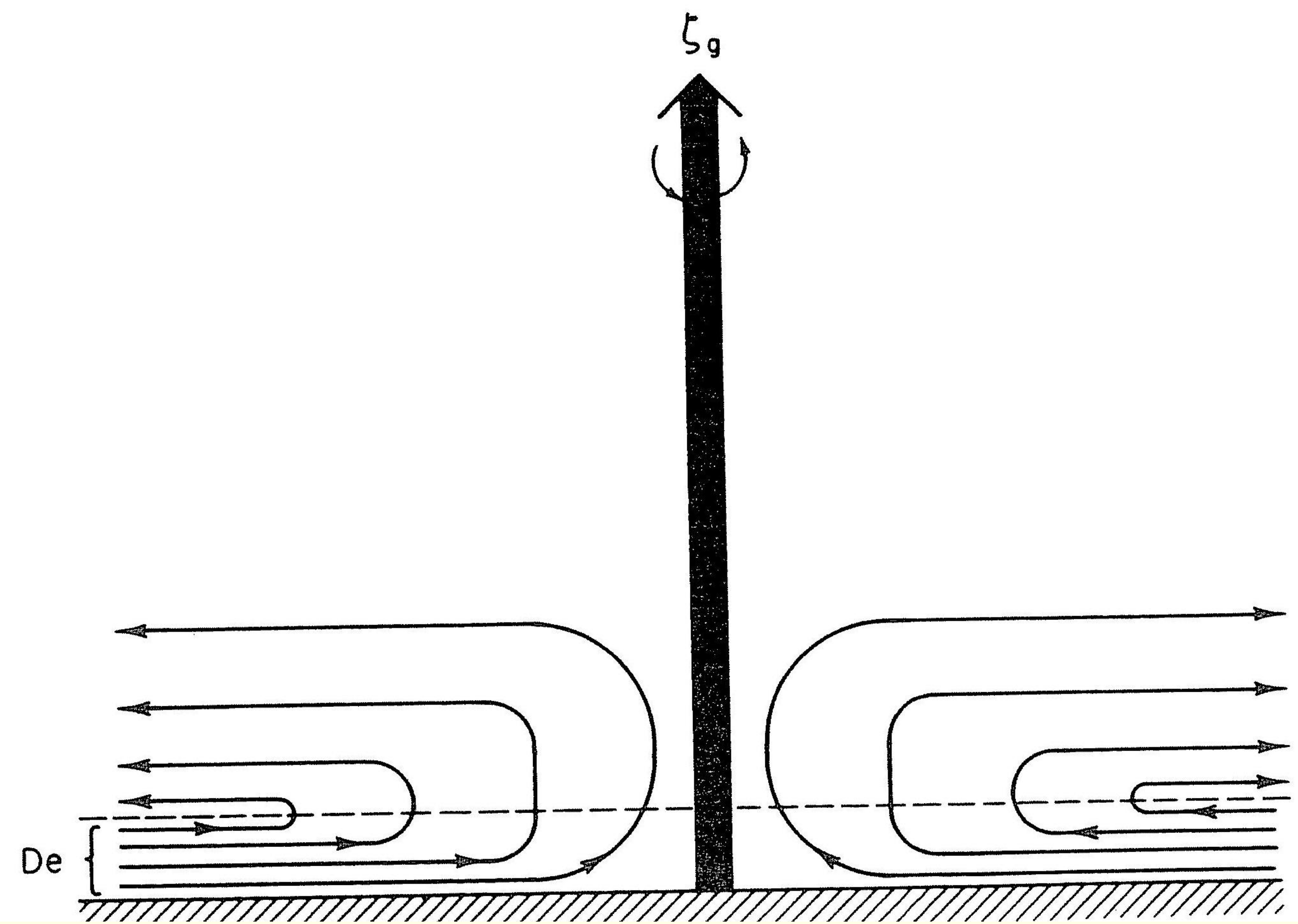
As a result, there is radial inflow near the bottom. By continuity of mass, this must result in upward motion near the centre.

Furthermore, outflow must occur in the fluid above the boundary layer.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The geostrophic vorticity is given, in cylindrical coordinates, by

$$\zeta_{\mathrm{g}} = \mathbf{K}\cdot\nabla\times\mathbf{V}_{\mathrm{g}} = \frac{1}{r}\left[\frac{\partial(rV_{\mathrm{g}})}{\partial r} - \frac{\partial U_{\mathrm{g}}}{\partial\theta}\right] = \frac{1}{r}\frac{\partial(\omega r^2)}{\partial r} = 2\omega$$

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$$V_{\mathrm{g}} = \omega\, r = 2\pi\,\mathrm{s}^{-1} \times 5\,\mathrm{cm} \approx 30\,\mathrm{cm\,s}^{-1}$$

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Exercise: Create a storm in a teacup:
Stir your tea (no milk) and observe the leaves.

Exercise:

- Calculate the mass influx through the sides of a cyclone.
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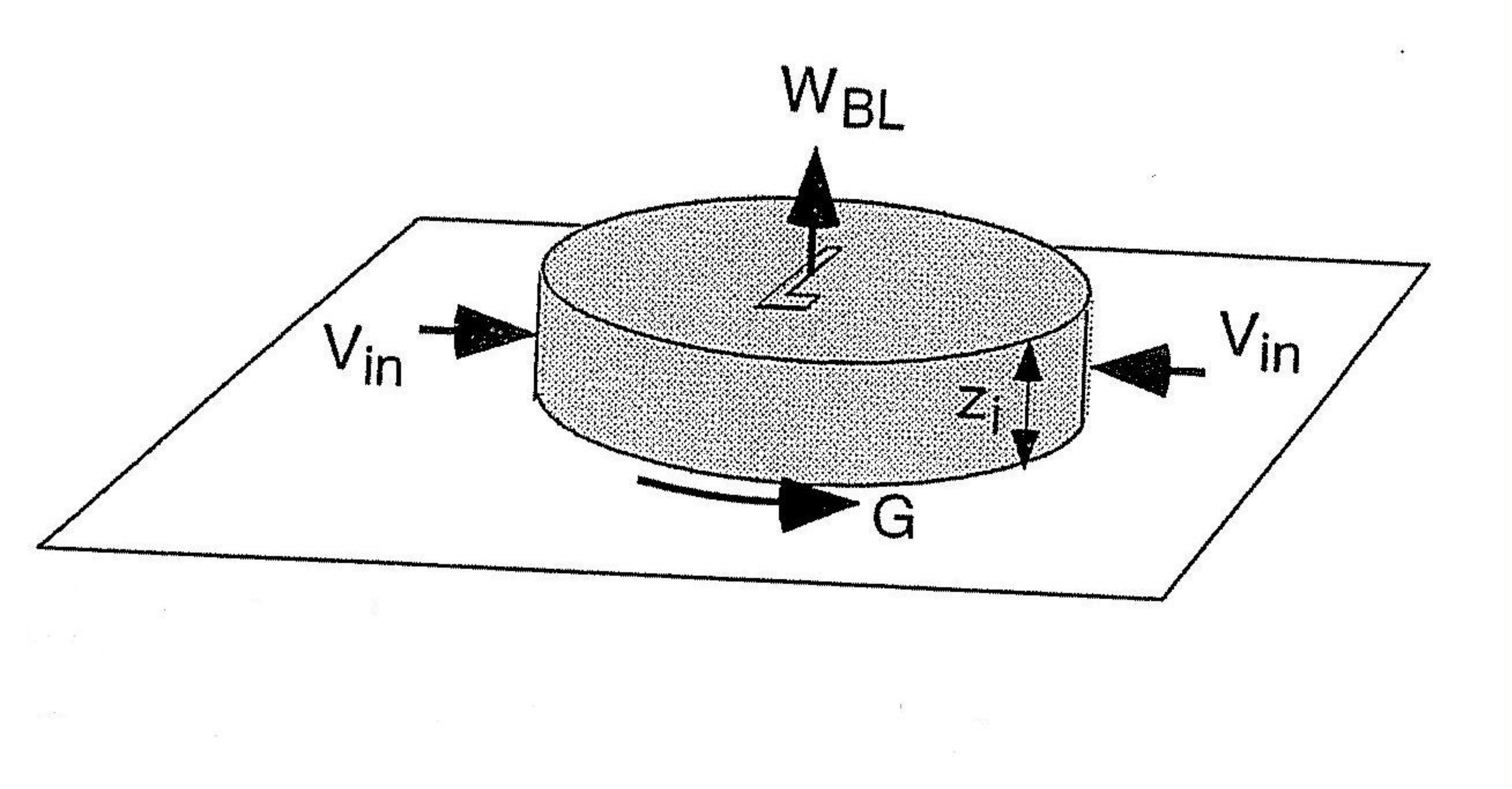

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From the Ekman solution, the mean inward velocity is

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For solid body rotation, $u_{\mathrm{g}} = \omega R$ and the geostrophic vorticity is $\zeta_{\mathrm{g}} = 2\omega$, so

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Assuming $w(H) = 0$ and substituting the Ekman pumping for $W(D)$ we get

$$\frac{d\zeta_{\rm g}}{dt} = -\frac{f}{(H - D)}\left(\frac{1}{2\pi}\right)D\zeta_{\rm g}$$

Assuming $H \gg D$, this is

$$\frac{d\zeta_{\mathrm{g}}}{dt} = -\frac{fD}{2\pi H}\,\zeta_{\mathrm{g}}$$

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The size of τ_{Ekman} may be estimated for typical values:

$$\tau_{\mathrm{Ekman}} = \frac{2\pi H}{fD} = \frac{2\pi \times 10^4}{10^{-4} \times 10^3} \approx 6 \times 10^5\,\mathrm{s}$$

which is about seven days.

We may compare this to the time-scale for eddy diffusion. The diffusion equation is

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$$\tau_{\mathrm{Diff}} = \frac{H^2}{K} \approx \frac{10^8}{5} = 2 \times 10^7\,\mathrm{s}$$

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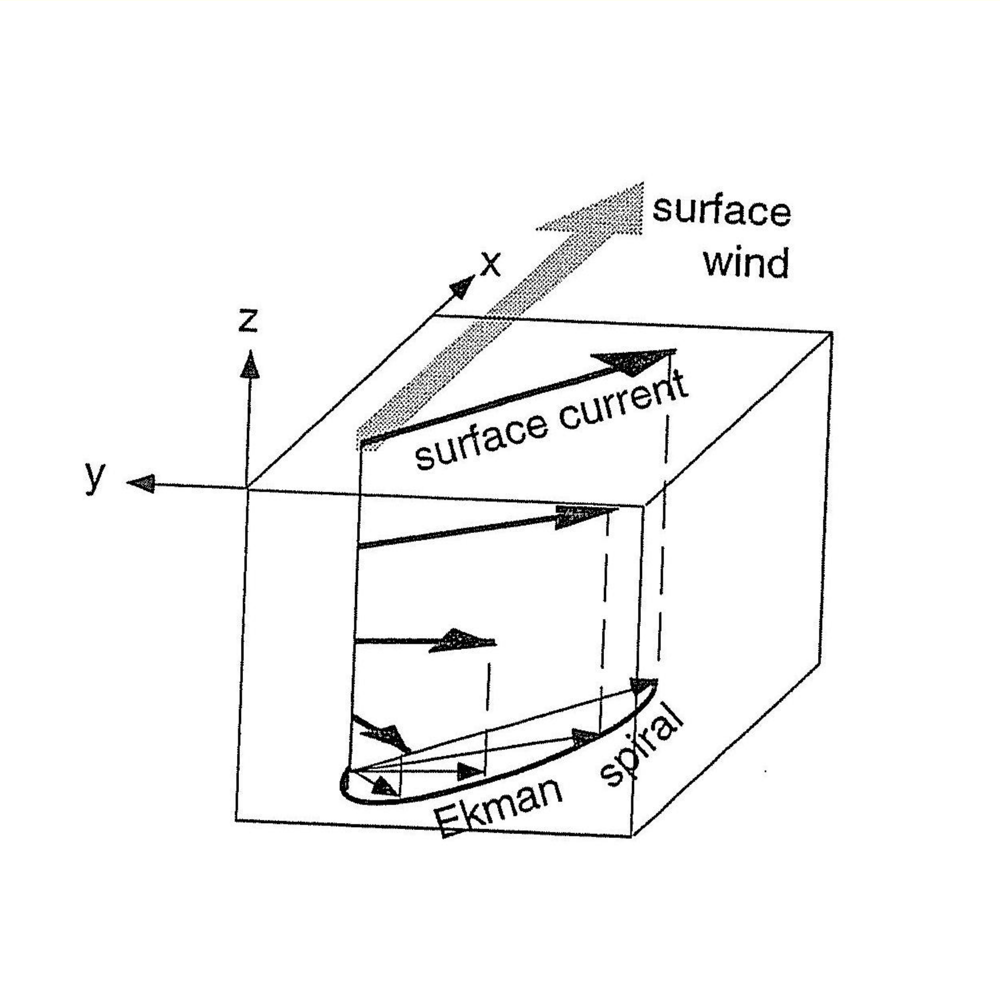
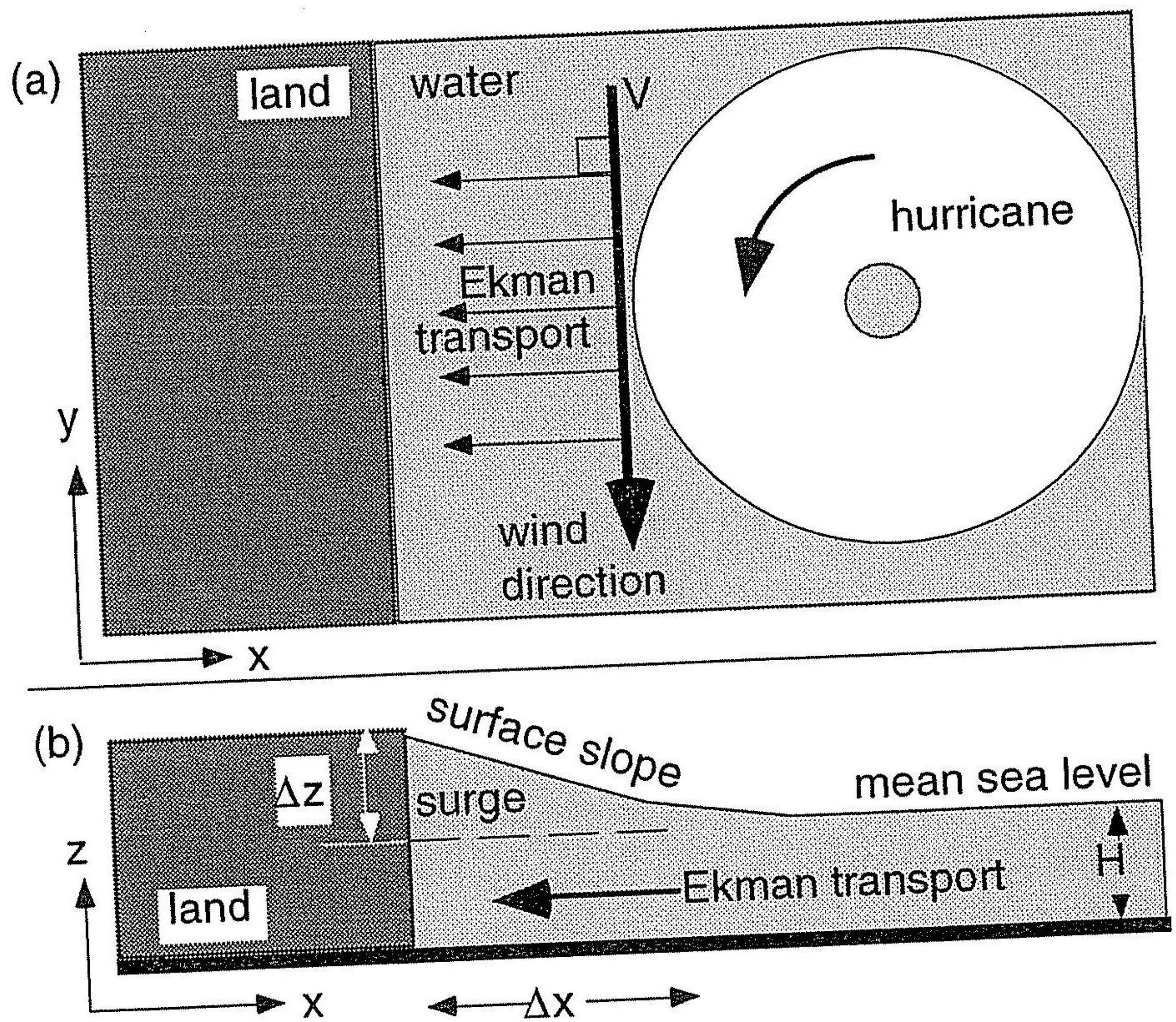
However, cumulonumbus convection can produce rapid transport of heat and momentum through the entire troposphere.

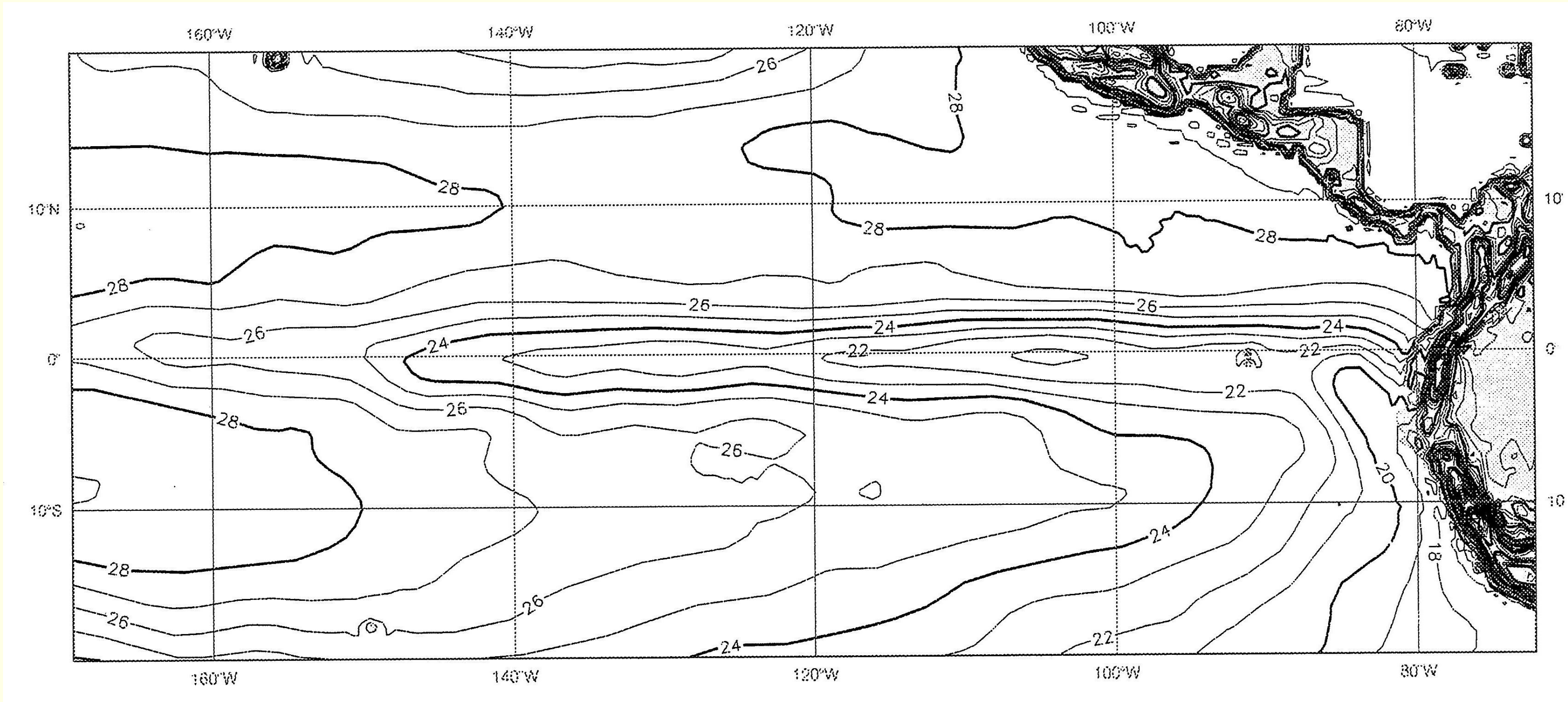
Ekman Layer in the Ocean



Ekman spiral in the ocean.


(a)
land
water
V
hurricane
Ekman
transport
y
wind
direction
x
(b)
surface slope
Δz
surge
mean sea level
z
H
Ekman transport
land
x
Δx

Typical La Niña Pattern



Mean sea surface temperature, eastern Pacific Ocean
5 September to 5 October, 1998.

End of §5.5