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The momentum equations will be taken in the form

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\frac{du}{dt} - fv + \frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}\left(\overline{w'u'}\right) &= 0 \\
\frac{dv}{dt} + fu + \frac{1}{\rho_0}\frac{\partial p}{\partial y} + \frac{\partial}{\partial z}\left(\overline{w'v'}\right) &= 0
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When the eddy fluxes are parameterized in terms of the mean flow, as indicated in the previous lecture, the momentum equations become

$$-fv + \frac{1}{\rho}\frac{\partial p}{\partial x} - K\frac{\partial^2 u}{\partial z^2} = 0$$

$$+fu + \frac{1}{\rho}\frac{\partial p}{\partial y} - K\frac{\partial^2 v}{\partial z^2} = 0$$

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where K and f may be assumed to be constant.

Defining $\gamma = \sqrt{f/2K}$ and assuming that the motion vanishes at $z = 0$ and tends to the zonal geostrophic value $\mathbf{V} = (u_g, 0)$ in the free atmosphere, derive the equations

$$
\begin{aligned}
u &= u_g(1 - e^{-\gamma z}\cos\gamma z) \\
v &= u_g e^{-\gamma z}\sin\gamma z
\end{aligned}
$$

corresponding to the Ekman spiral.

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We define the components of the geostrophic velocity as

$$u_G = -\frac{1}{f\rho}\frac{\partial p}{\partial y} \qquad\qquad v_G = +\frac{1}{f\rho}\frac{\partial p}{\partial x}$$

and the corresponding complex geostrophic velocity as

$$w_G = u_G + iv_G$$

Now we may write the equations of motion as

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Now multiply the first equation by i and subtract it from the second:

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We re-write this as

$$\frac{\partial^2 w}{\partial z^2} - \left(\frac{if}{K}\right) w = -\left(\frac{if}{K}\right) w_G$$

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2. **Find a *Complementary Function* (CF), a general solution of the homogeneous part of the equation.**

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2. **Find a *Complementary Function* (CF), a general solution of the homogeneous part of the equation.**

***Particular Integral*: Clearly, one solution of the inhomogeneous equation is obtained by assuming that w is independent of z. This reduces the equation to**

$$-\left(\frac{if}{K}\right) w = -\left(\frac{if}{K}\right) w_G$$

with the solution $w = w_G$.

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Thus, there are two possible values of λ:

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Now we have

$$\lambda_+ = (1+i)\gamma \qquad \textbf{and} \qquad \lambda_- = (-1-i)\gamma$$

The general solution of the homogeneous equation is

$$\begin{aligned} w &= A \exp \lambda_+ z + B \exp \lambda_- z \\ &= A \exp(1+i)(\gamma z) + B \exp(-1-i)(\gamma z) \\ &= A \exp(\gamma z) \exp(i\gamma z) + B \exp(-\gamma z) \exp(-i\gamma z) \end{aligned}$$

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The term multiplied by A grows exponentially with z and so must be rejected. The physically acceptable solution is thus

$$w = B\exp(-\gamma z)\exp(-i\gamma z)$$

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Expanding this into real and imaginary parts, we have

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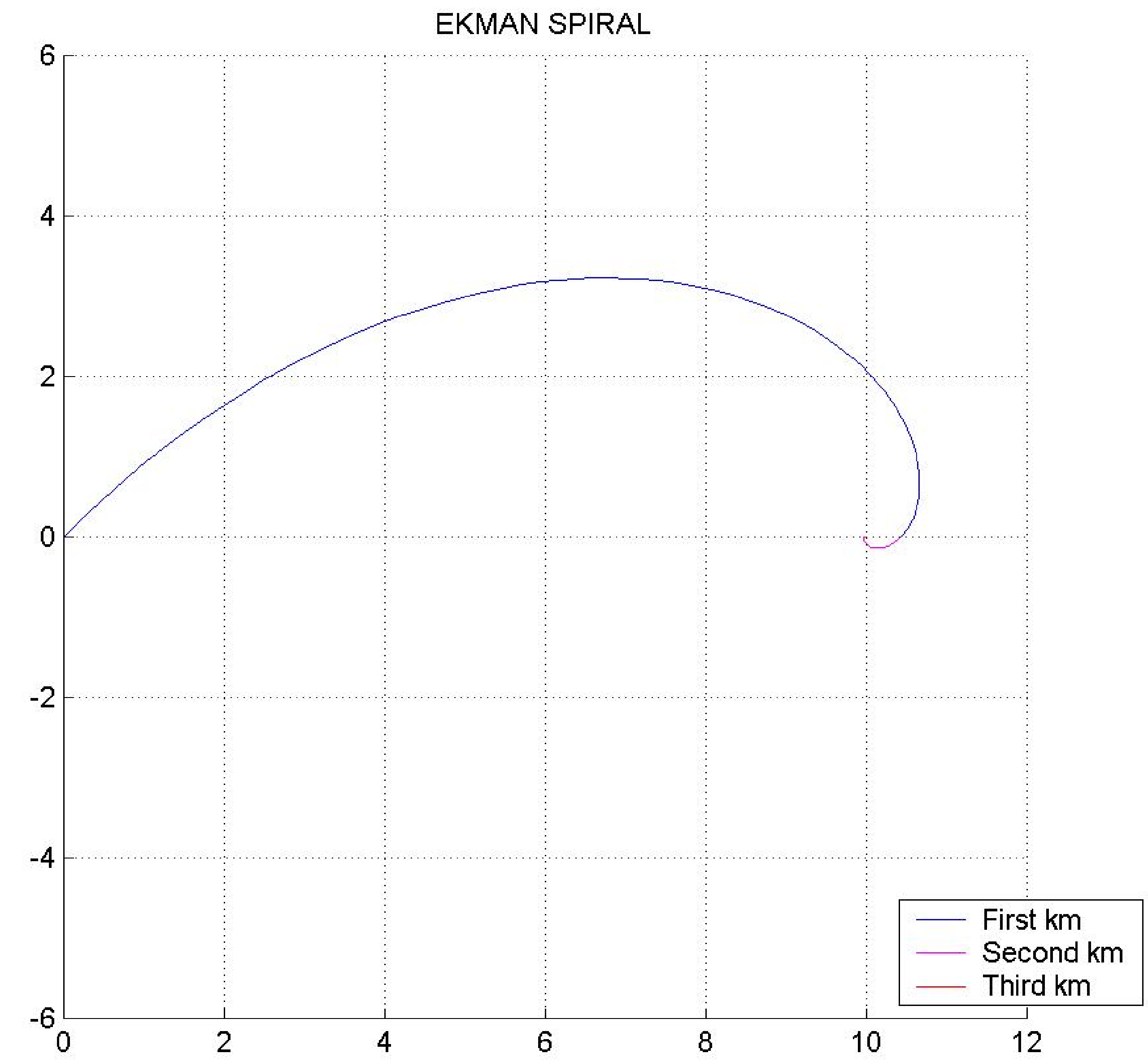
For simplicity, we now assume that the geostrophic wind is purely zonal, so that $v_G = 0$. Then, separating the real and imaginary components of w, we have

$$\begin{aligned} u &= u_G\,[1 - \exp(-\gamma z)\cos(\gamma z)] \\ v &= u_G\,[\quad + \exp(-\gamma z)\sin(\gamma z)] \end{aligned}$$



Horizontal axis: u. Vertical axis: v. Geostrophic wind: $u_G = 10\,\mathrm{m\,s^{-1}}$.

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- **The flow is *super-geostrophic* at this point.**
- **The height where this occurs may be taken as the *effective height* of the Ekman layer. The wind is close to geostrophic above this height.**

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The *effective height* is $z_0 = \pi/\gamma$. With $f = 10^{-4}\,\mathbf{s}^{-1}$ and $K = 10\,\mathbf{m}^2\mathbf{s}^{-1}$ we have

$$z_0 = \frac{\pi}{\gamma} = \pi\sqrt{\frac{2K}{f}} = \pi\sqrt{\frac{2 \times 10}{10^{-4}}} \approx 1400\,\mathrm{m}$$

Thus, the effective depth of the Ekman boundary layer is about $1.4\,\mathbf{km}$.

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- **The solution is then called a** ***modified Ekman spiral*****.**

MatLab Exercise:

Write a program to calculate the wind speed as a function of altitude. Assume the values $f = 10^{-4}\,\mathbf{s}^{-1}$ and $K = 10\,\mathbf{m}^2\mathbf{s}^{-1}$.

End of §5.4