# The Molecular Viscous Force

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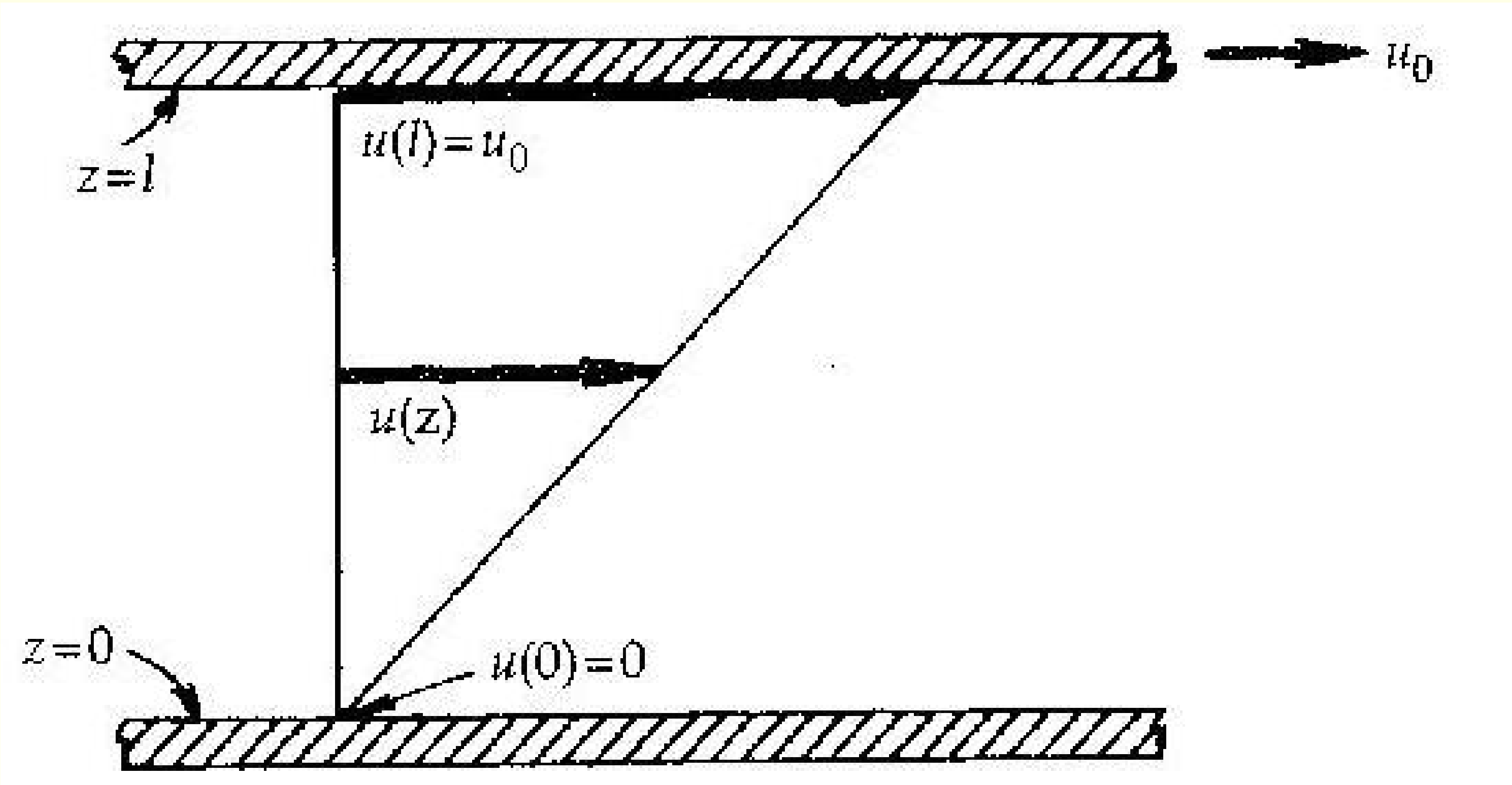
**To gain an understanding of this, we consider a very simple case, with a flow confined between two solid plates:**



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**A force is required to keep the upper plate in motion. Experiments show that this force is:**

- **proportional to the area $A$ of the plates**
- **proportional to the velocity $u_0$**
- **inversely proportional to the distance $\ell$ between the plates.**

**Thus, we may write**

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**We may assume the speed $u$ of the fluid varies linearly across the layer from $z = 0$ to $z = \ell$.**

**Thus, we may write the force as**

$$F = \mu \frac{A\, u_0}{\ell} = \mu A \left( \frac{u(\ell) - u(0)}{\ell} \right) = \mu A \frac{\partial u}{\partial z}$$

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**The subscripts of $\tau_{zx}$ denote that it is the component in the $x$-direction due to velocity shear in the $z$-direction.**

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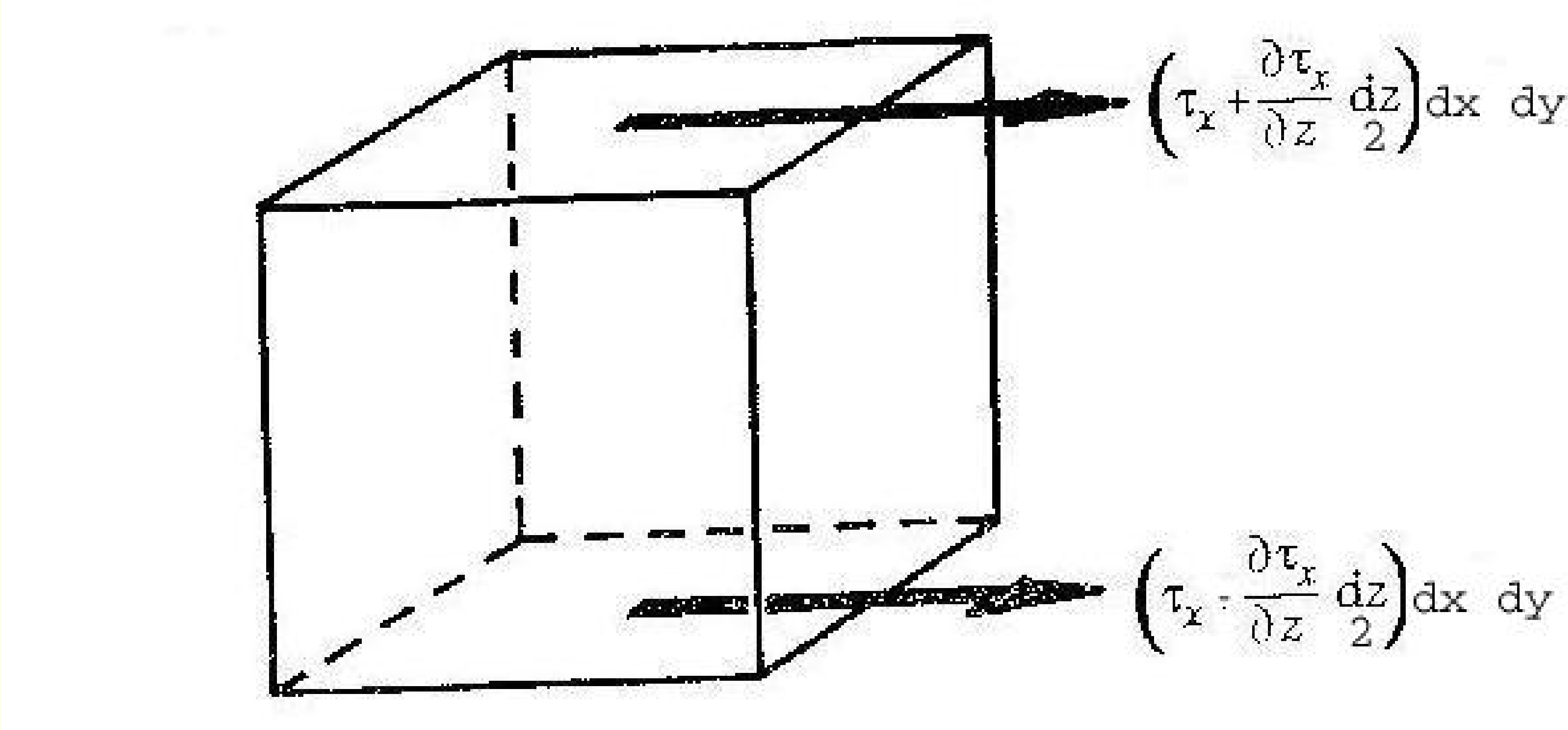
**In a more general case of unsteady (accelerated) flow, we may calculate the viscous force by considering an element of fluid with sides $\delta x$, $\delta y$ and $\delta z$.**

**Let us assume the shearing stress at the centre of the element is $\tau_{zx}$.**
**Then, the stress at the upper surface may be written**

$$\left[\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right]$$

**The stress at the lower surface may be written**

$$-\left[\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right]$$


$\left(\tau_x + \frac{\partial \tau_x}{\partial z}\frac{dz}{2}\right)dx\ dy$
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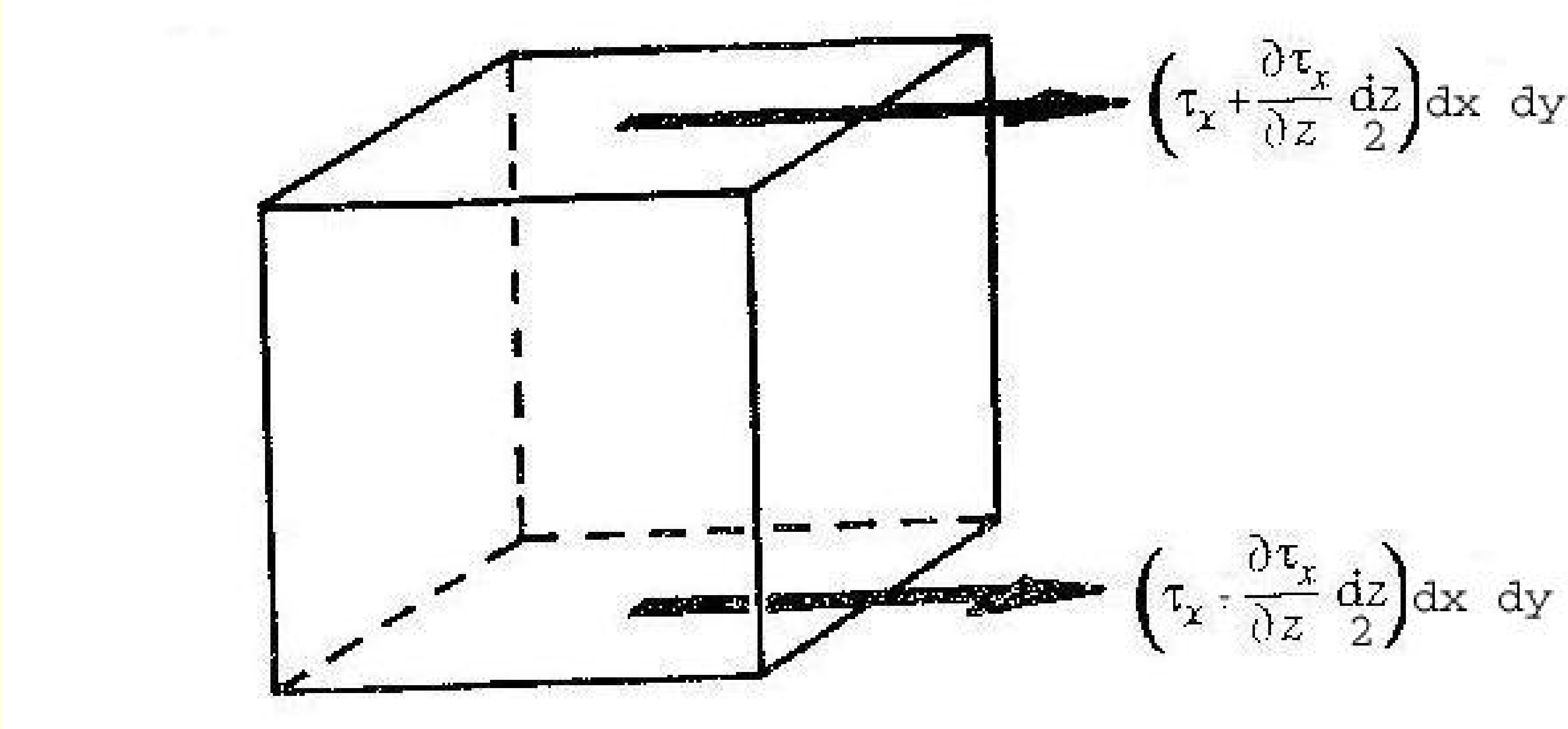


**The net force is the sum of viscous forces on the upper and lower faces of the element:**

$$\left[\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right]\delta x \delta y - \left[\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right]\delta x \delta y = \left[\frac{\partial \tau_{zx}}{\partial z}\right]\delta x \delta y \delta z$$

**To obtain the force per unit mass, we divide by the mass $\rho\delta x\delta y\delta z$ of the element:**

$$F = \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} = \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)$$

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**If the *dynamic viscosity coefficient* $\mu$ is constant, we may simplify this to**

$$F = \frac{\mu}{\rho}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \nu\left(\frac{\partial^2 u}{\partial z^2}\right)$$

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**For standard atmospheric conditions at sea level, the kinematic viscosity coefficient has the value $\nu = 1.5\times10^{-5}\,\mathbf{m}^2\,\mathbf{s}^{-1}$.**

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**More generally, the wind may vary in all directions, and the viscous force will have three components, which may be written**

$$F_x = \nu \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right] = \nu \nabla^2 u$$

$$F_y = \nu \left[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right] = \nu \nabla^2 v$$

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**Away from this *molecular boundary layer*, momentum is transferred primarily by turbulent eddy motions. These are considered next.**

# End of §5.2