

## Introduction

- These 12 lectures form the introductory part of the course on Numerical Weather Prediction for the M.Sc. in Meteorology at UCD. The aims of this part of the NWP course are to

1. introduce and discuss the analytic/exact solution of the **linear advection equation**

$$\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = 0$$

where  $v(x, t)$  is given and we wish to solve for  $u(x, t)$  starting from some initial condition (as we shall see this equation describes the advection of the function  $u$  at speed  $v$ ),

2. introduce the **finite difference method** for solving the advection equation numerically,
3. discuss the issue of **numerical stability** and the **Courant Friedrich Lewy (CFL)** condition,
4. extend the above methods to non-linear problems such as the **inviscid Burgers equation**

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$$

This is similar to the advection equation in appearance but has a crucial difference in that the advection speed is now equal to  $u(x, t)$ . It is the left hand side of equation ?? which occurs in the primitive equations of gas dynamics and its numerical solution is critical in the study of NWP.

- Finally we will also write programmes, and do exercises, for the methods introduced in these lectures. This will be done using the MATLAB language.

## Textbooks

- Although the lecture notes and programmes will be self contained there are chapters in some well known textbooks that provide useful further material and examples.
  1. *Atmospheric Modelling, Data Assimilation and Predictability* by E. Kalnay. Chapter 3 discusses numerical discretisation.
  2. *Numerical Methods for Physics* by A. Garcia (Prentice Hall). Good mix of numerical methods, applications and Matlab programmes. Chapter 1 is good for MATLAB and chapter 6 discusses the advection equation.
  3. *Numerical Solution of Partial Differential Equations* by K.W. Morton and D.F. Mayers (Cambridge University Press). Very good on the numerical analysis of pde's. More formal and mathematical than the previous text, chapter 4 is concerned with advection problems.

4. *Numerical Methods for Conservation Laws* by R.J. LeVeque (Birkhauser). Not for the faint-hearted - intended as further reading ! But accessible introductory chapters.

## 1. Basic Definitions

- We begin with some basic definitions most of which you will have come across already in a different context. Consider a **gas moving in one dimension**, along the  $x$  axis, and let  $\rho(x, t)$  be the mass density at point  $x$  and time  $t$ . If  $u(x, t)$  is the bulk gas flow velocity then  $\rho u$  is the mass flux at  $x$  and  $t$ . If matter is neither created nor destroyed then the mass density at a point can only change if there is a gradient in the mass flux,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

which is an example of a **conservation law** and the type of equation we are interested in solving. In Meteorology we are, of course, interested in three dimensional flow in which case this law takes the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

- We can also write down two further equations describing momentum conservation and energy conservation. The former case describes how the fluid accelerates as a result of a pressure gradient (in the absence of viscosity) and can be written in the form

$$\frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + P) = 0$$

where  $P$  is the pressure. We can combine this with the mass conservation equation and re-write it as

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial P}{\partial x} \quad (1)$$

which contains the Lagrangian derivative of  $u$  and the pressure gradient term which you have already seen in the primitive equations. Finally, for completeness, the **energy conservation equation**,

$$E_t + (v(E + P))_x = 0$$

where  $E$  is the energy density. An **equation of state - the ideal gas law** is also needed to relate  $E$ ,  $P$  and  $\rho$ .

- All of the above partial differential equations (pde's) have the same **conservative form**,

$$\frac{\partial}{\partial t}(\text{Density}) + \frac{\partial}{\partial x}(\text{Flux}) = 0$$

We want to study methods of solving such equations when exact or analytic methods are not available.

- Some numerical methods solve the equations in their pure conservative form (i.e. in terms of densities such as the momentum density  $\rho u$  and so on). Once the densities are known they then return to the **primitive variables** such as  $\rho$  and  $v$ . In these lectures we will concentrate on solving for the primitive variables directly, for example equation 1.

## 1.2 Linear Advection Equation

- Physically equation 1 says that as we follow a fluid element (the Lagrangian time derivative), it will accelerate as a result of the local pressure gradient and this is one of the most important equations we will need to solve.
- However, we will begin by looking at a far simpler equation which is known as the **linear advection equation** for the quantity  $u(x, t)$

$$\frac{\partial u}{\partial t} + v(x, t) \frac{\partial u}{\partial x} = 0 \quad (2)$$

where  $v(x, t)$  is known. We will also have to supplement this equation with an **initial condition**,  $u(x, 0)$  and, if necessary, **boundary conditions** (we will discuss these later).

Physically this equation says that if we start from any point and time and subsequently always move with a speed given by  $v(x, t)$  then  $u$  will never change from its initial value. To see this consider the change in  $u$  in moving from point  $(x, t)$  to point  $(x + dx, t + dt)$ ,

$$du = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dx$$

Therefore, if we move along any curve  $x(t)$  in the  $(x, t)$  plane then, along that curve,  $u$  will change at the rate

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \left( \frac{dx}{dt} \right) \frac{\partial u}{\partial x}$$

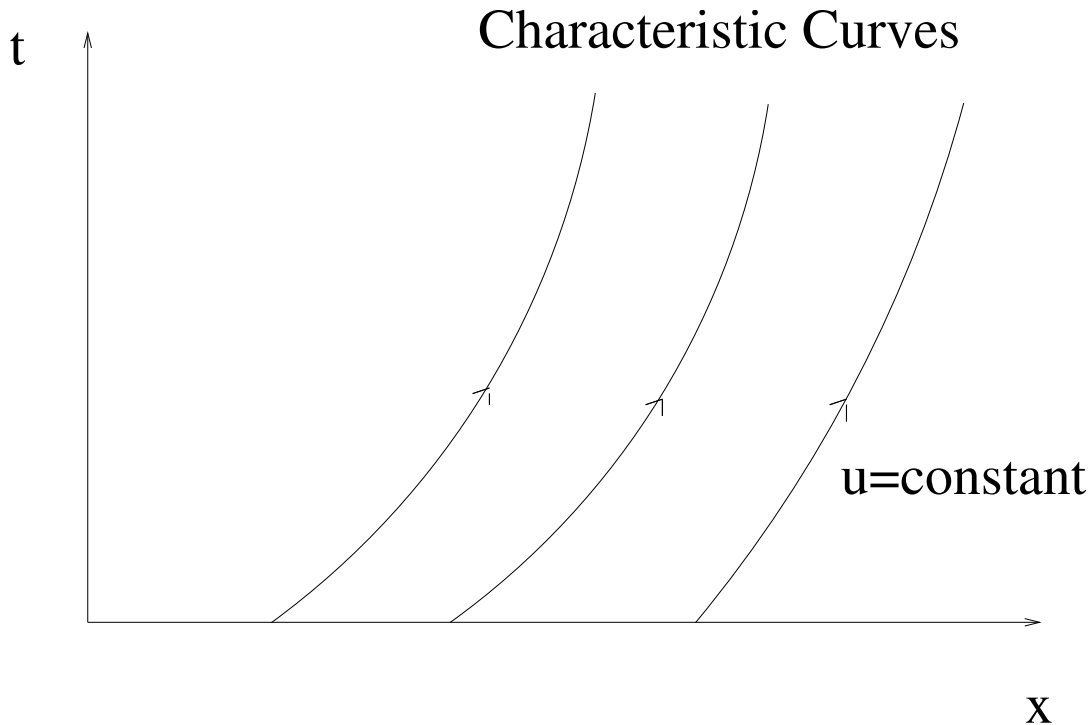
There are two contributions to the right hand side. The first is the explicit dependence on  $t$  (what we would see if we stood still) and the second is the change we see by our movement along the  $x$  axis.

- For the problem given by equation 2 there are special curves called **characteristics** defined by

$$\frac{dx}{dt} = v(x, t) \quad (3)$$

along which  $u$  is constant,

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + v(x, t) \frac{\partial u}{\partial x} = 0$$



- Therefore, once we have an **initial condition**

$$u(x, 0) = u_0(x)$$

we can find the characteristic curve that originates from any point  $x_*$  on the  $x$  axis, i.e. solve the problem

$$\frac{dx}{dt} = v(x, t) \quad \text{where} \quad x(t) = x_*$$

and along that curve  $u(x, t) = u(x_*, 0) = u_0(x_*)$ ; i.e. to find the solution at  $(x, t)$  we need to find where the characteristic passing through that point hits the  $x$  axis where the value of  $u$  is simply set by the initial condition.

Therefore, the method of characteristics reduces the problem of solving a partial differential equation to the solution of an ordinary differential equation.

## Examples

1. Solve the advection equation where  $v$  is a constant

$$\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = 0$$

with initial condition  $u(x, 0) = \exp(-x^2/L^2)$ , a Gaussian profile.

A characteristic curve, along which  $u(x, t)$  is constant, is the solution to

$$\frac{dx}{dt} = v \quad \Rightarrow \quad x = x_* + vt$$

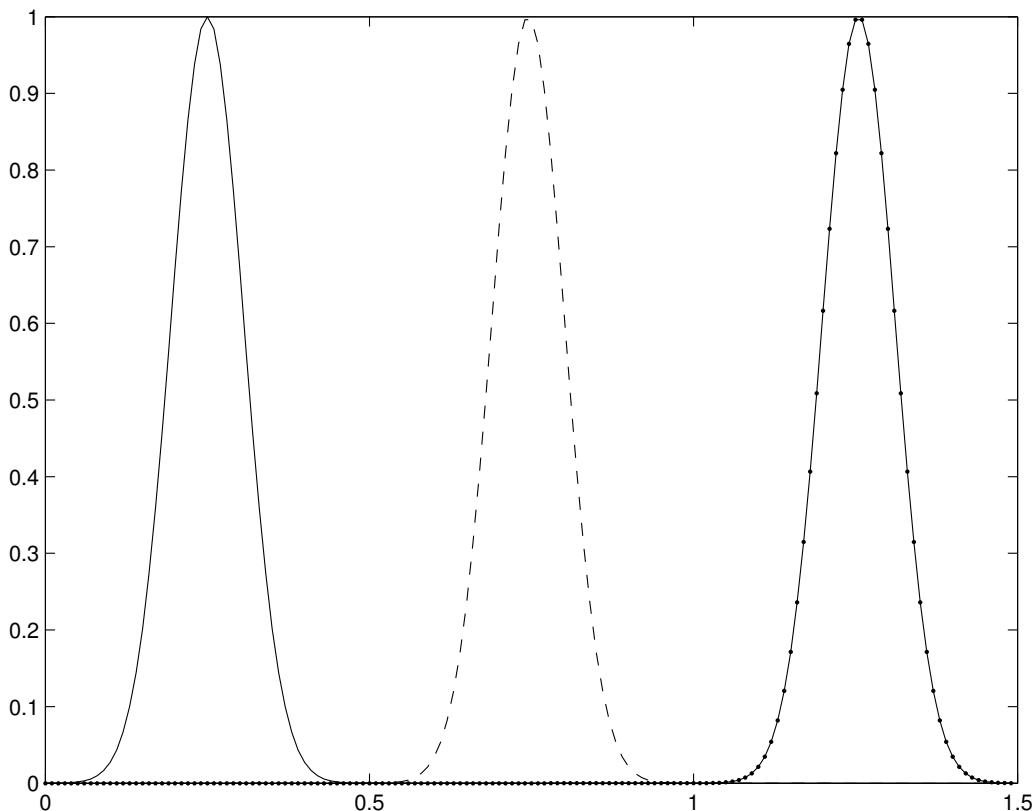
so that the characteristics are straight lines on the  $(x, t)$  plane. Therefore the solution is

$$u(x, t) = u(x_*, 0) = u_0(x - vt)$$

which describes the the initial shape moving with velocity  $v$ . For the initial shape of a Gaussian profile then at later times we have

$$u(x, t) = \exp\left(-\frac{(x - vt)^2}{L^2}\right)$$

the Gaussian at time  $t$  is now centered at  $x = vt$ . In the diagram below we plot the Gaussian for three successive time intervals, from left to right, and it can be seen that the shape of that profile is perfectly preserved.



2. Solve the more difficult problem, again with a Gaussian initial profile,

$$\frac{\partial u}{\partial t} + \frac{1+x^2}{1+2xt+2x^2+x^4} \frac{\partial u}{\partial x} = 0$$

Characteristic curve is solution to

$$\frac{dx}{dt} = \frac{1+x^2}{1+2xt+2x^2+x^4} \quad \Rightarrow \quad t = (x-x_*)(1+x^2)$$

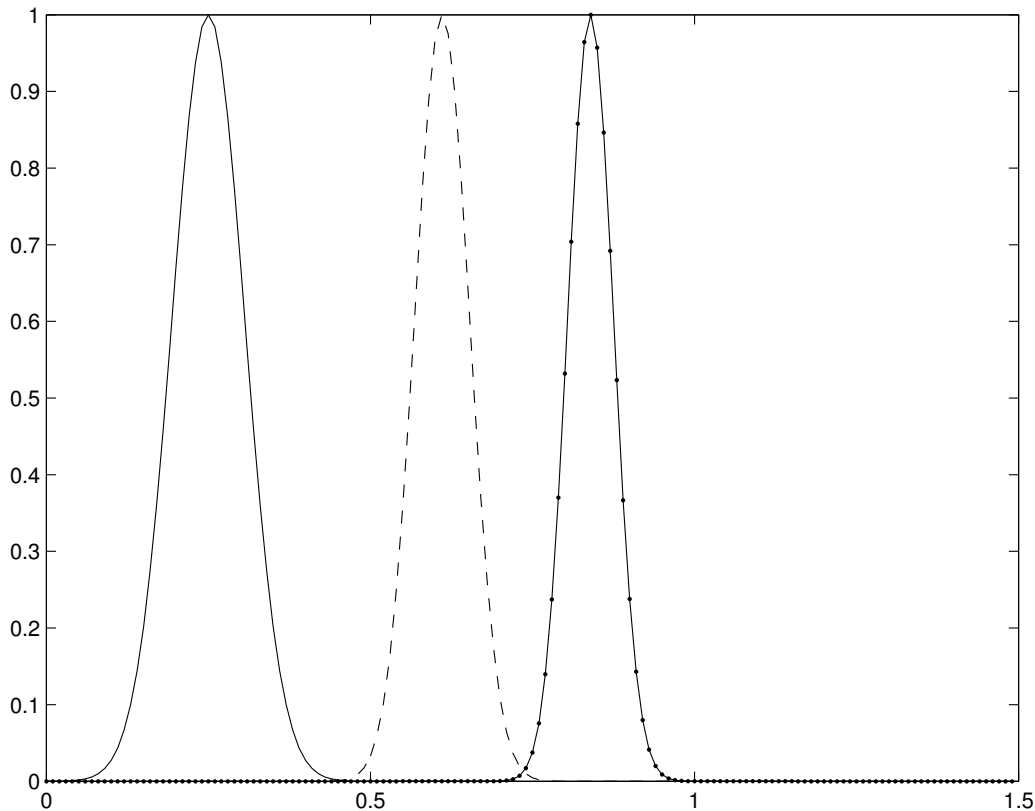
(check this !) which gives the solution to the pde in terms of the initial condition

$$u(x, t) = u(x_*, 0) = u_0 \left( x - \frac{t}{1+x^2} \right)$$

Again if the initial shape is  $u(x, 0) = \exp(-x^2/L^2)$  then

$$u(x, t) = \exp \left[ -\frac{1}{L^2} \left( x - \frac{t}{1+x^2} \right)^2 \right]$$

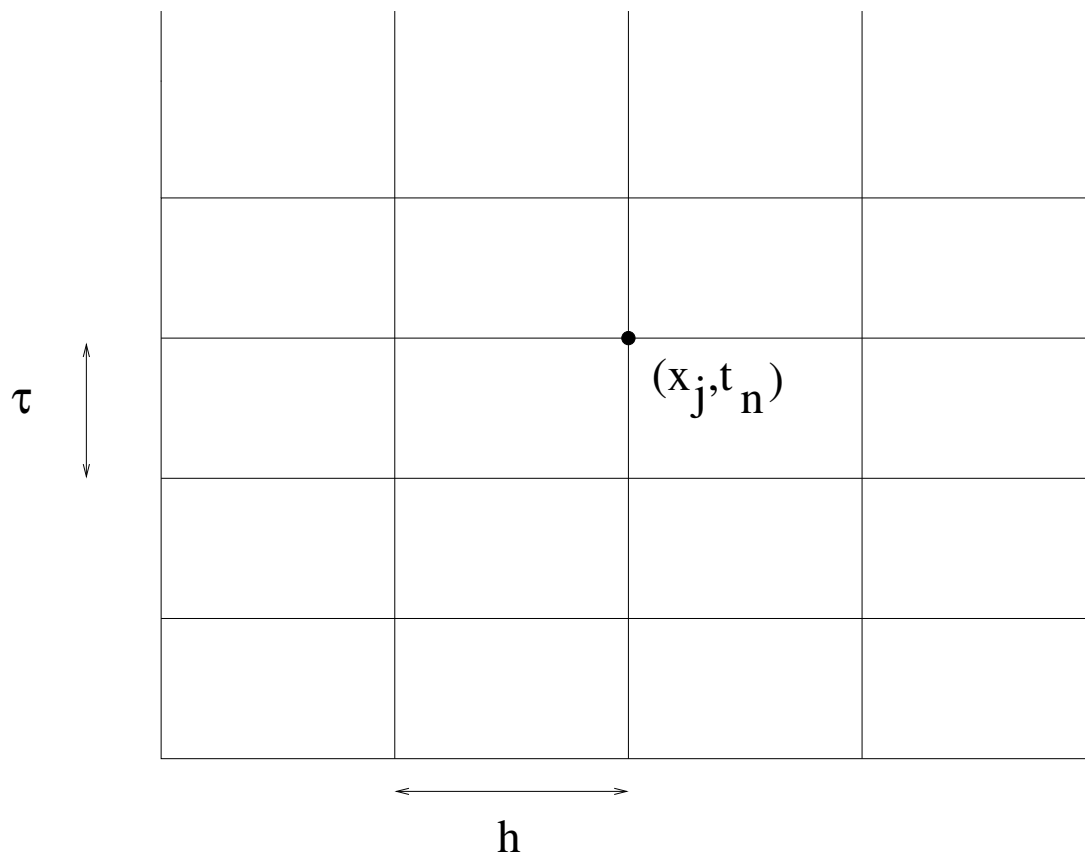
In this case the initial shape, although it moves to the right, is not preserved (as can be seen below) since the speed,  $v(x, t)$ , is not constant.



### 1.3 Finite difference methods for linear advection

- How could we solve the linear advection equation if  $v(x, t)$  were too complicated to use the analytic method of characteristics, i.e. **how do we solve the linear advection equation numerically** ?
- We begin by replacing the continuous  $(x, t)$  plane by a **grid** where we have a uniform spacing,  $h$ , along the  $x$  axis and timestep,  $\tau$ , on the  $t$  axis.

$$\begin{aligned}x_j &= x_0 + jh & j &= 0, 1, \dots, J \\t_n &= t_0 + n\tau & n &= 0, 1, \dots, N\end{aligned}$$



If we want to solve a partial differential equation for  $u(x, t)$  on a computational grid, we only consider the values of  $u$  at discrete points.

$$u(x, t) \rightarrow u(x_0 + jh, t_0 + n\tau) \equiv u_j^n$$

**i.e.**  $u_j^n$  = the numerical value for  $u$  at the  $j$ th spatial point and  $n$ th temporal point.

- We say that a numerical scheme is **convergent** if  $u_j^n \rightarrow u(x_j, t_n)$  as  $h \rightarrow 0$  and  $\tau \rightarrow 0$ ; i.e. if we make the grid progressively more refined we want the numerical solution to tend to the analytical one. **Otherwise the numerical scheme is useless !!**

- Several numerical methods can be constructed for equation 2 and we discuss these below. Although this equation has a simple analytic solution the numerical solution is not trivial - it is actually one of the most difficult problems we will have to solve numerically !

### Finite Differences

- How do we express the derivatives of  $u(x, t)$  in terms of  $u_j^n$  ?
- Before we answer that question we will consider a **function of one variable,  $f(x)$** . We introduce a grid on the  $x$  axis of points equally spaced by the distance  $h$ ;  $\dots, x_{j-1}, x_j, x_{j+1}, \dots$ . At these points the **numerical approximations are denoted by  $f_j$  etc.** and the **analytic values are given by  $f(x_j)$** .
- With  $f'_j$  the first derivative of  $f(x)$  w.r.t.  $x$  at point  $x_j$  we can write down the Taylor expansions for  $f(x)$  to either side of  $x_j$

$$\begin{aligned} f(x_{j+1}) &= f(x_j) + hf'_j + \frac{h^2}{2}f''_j + O(h^3) \\ f(x_{j-1}) &= f(x_j) - hf'_j + \frac{h^2}{2}f''_j - O(h^3) \end{aligned}$$

Subtracting these Taylor series gives,

$$f(x_{j+1}) - f(x_{j-1}) = 2hf'_j + O(h^3) \quad \Rightarrow \quad f'_j = \frac{f(x_{j+1}) - f(x_{j-1})}{2h} + O(h^2)$$

so that the first derivative can be replaced by the difference between  $f$  to either side, divided by  $2h$ , which produces an error proportional to  $h^2$ .

**Computationally** we can use this centered difference to **approximate the first derivative on our grid** and this is called the **centered difference approximation to the first derivative**

$$f'_j = \frac{f_{j+1} - f_{j-1}}{2h} \quad (4)$$

which is second order accurate in  $h$ .

We could, of course, have used each original Taylor expansion separately to approximate the derivative,

$$\begin{aligned} f'_j &= \frac{f_{j+1} - f_j}{h} && \text{(Forward difference)} \\ f'_j &= \frac{f_j - f_{j-1}}{h} && \text{(Backward difference)} \end{aligned}$$

which are each first order accurate in  $h$ .



- We can also get an approximation for the second derivative by adding the Taylor series to get

$$f_{j+1} + f_{j-1} = 2f_j + h^2 f_j'' + O(h^4)$$

which gives the **centered difference approximation to the second derivative** which is second order accurate

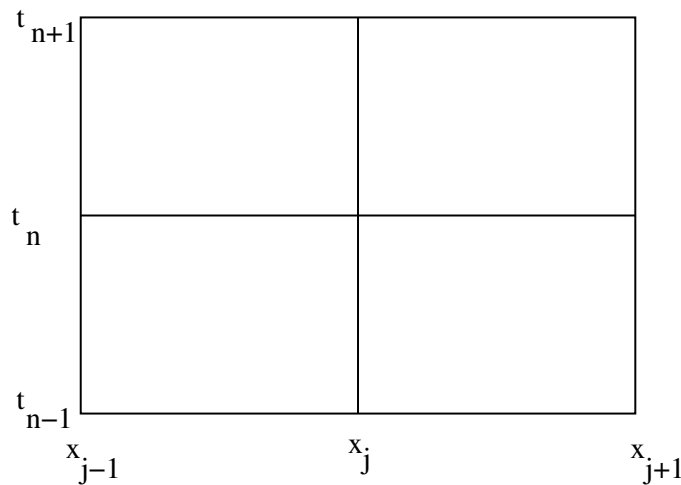
$$f_j'' = \frac{f_{j+1} - 2f_j + f_{j-1}}{h^2} \quad (5)$$

This expression may become a little clearer if written in the form

$$f_j'' = \frac{(f_{j+1} - f_j)/h - (f_j - f_{j-1})/h}{h}$$

- How do we generalise these difference formulae to a function of two variables ?

We now have a two-dimensional grid,  $u_j^n \approx$  value of  $u(x_j, t_n)$ .



At the  $n$ th time-step (i.e. at  $t_n$  we can use the formulae for  $\frac{df}{dx}$  and  $\frac{d^2f}{dx^2}$  to determine  $\frac{\partial u}{\partial x}$  and  $\frac{\partial^2 u}{\partial x^2}$ :

$$(1) \quad \left(\frac{\partial u}{\partial x}\right)_{j,n} = \frac{u_{j+1}^n - u_{j-1}^n}{2h} + O(h^2)$$

$$(2) \quad \left(\frac{\partial^2 u}{\partial x^2}\right)_{j,n} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} + O(h^2)$$

[In each of these formulae the index  $n$  does not change:  $t$  is constant]

Similarly, at the  $j$ th spatial position  $x_j$  we can determine  $\frac{\partial u}{\partial t}$  and  $\frac{\partial^2 u}{\partial t^2}$ :

$$(1) \left( \frac{\partial u}{\partial t} \right)_{j,n} = \frac{u_j^{n+1} - u_j^{n-1}}{2\tau} + O(\tau^2)$$

$$(2) \left( \frac{\partial^2 u}{\partial t^2} \right)_{j,n} = \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\tau^2} + O(\tau^2)$$

[Now the index  $j$  does not change:  $x$  is constant].

### Forward Time Centered Space (FTCS) method

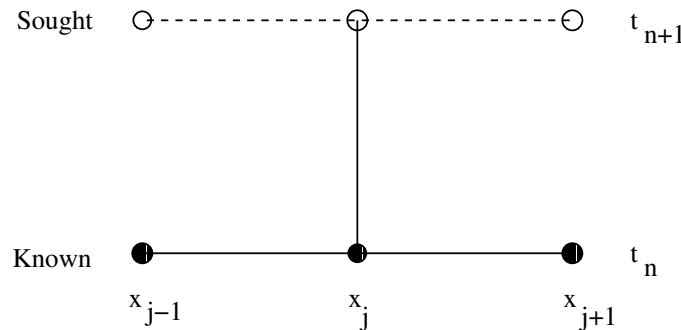
- Using forward time (because we must start from some initial condition) and centered space (because it's accurate) differencing on equation 2 gives,

$$\frac{u_j^{n+1} - u_j^n}{\tau} = -v_j^n \frac{u_{j+1}^n - u_{j-1}^n}{2h}$$

so that if we know all of the values at time  $t_n$  we can calculate all of the values at  $t_{n+1}$  from

$$u_j^{n+1} = u_j^n - \frac{\nu}{2} (u_{j+1}^n - u_{j-1}^n) \quad (\text{FTCS Method}) \quad (6)$$

where  $v_j^n \equiv v(x_j, t_n)$  and  $\nu = v_j^n \tau / h$ . This is known as the **FTCS method for the advection equation**. It is completely useless !!



- This method is applied in FTCS.m but before showing the results we need boundary conditions. If  $j = 1$ , the FTCS scheme gives

$$u_1^{n+1} = u_1^n - \frac{\nu\tau}{2h} (u_2^n - u_0^n)$$

but if  $j = 1$  is the leftmost point then  $u_0^n$  doesn't exist !

We need to implement a boundary condition when  $j = 1$  and  $j = N$ . In this case we impose **periodic boundary conditions**; as the wave is advected off the grid to the right, we would like it to re-appear on the left. We do this by setting  $u_0^n \rightarrow u_N^n$ , i.e.

$$u_1^{n+1} = u_1^n - \frac{v\tau}{2h} (u_2^n - u_N^n)$$

(i.e. we imagine that just to the left of  $j = 1$ , the wave amplitude is given by  $u_N^n$ ).

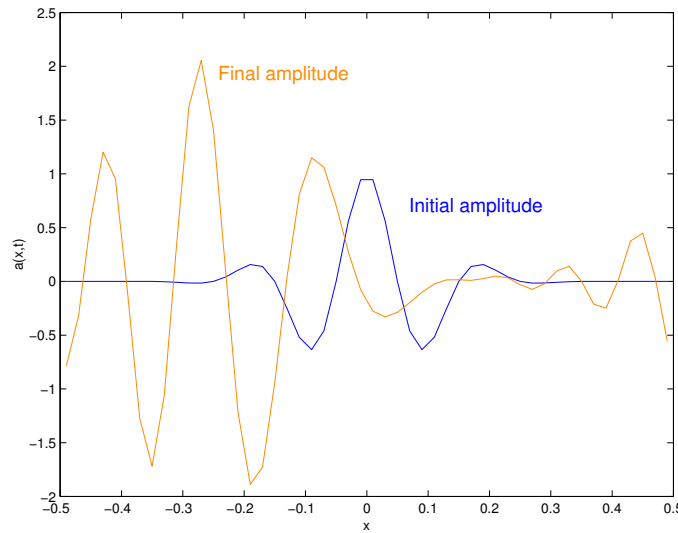
Similarly,  $u_N^{n+1} = u_N^n - \frac{v\tau}{2h} (u_1^n - u_{N-1}^n)$  As the wave disappears off the right hand side of the grid it will re-appear on the left

As an initial condition we take,

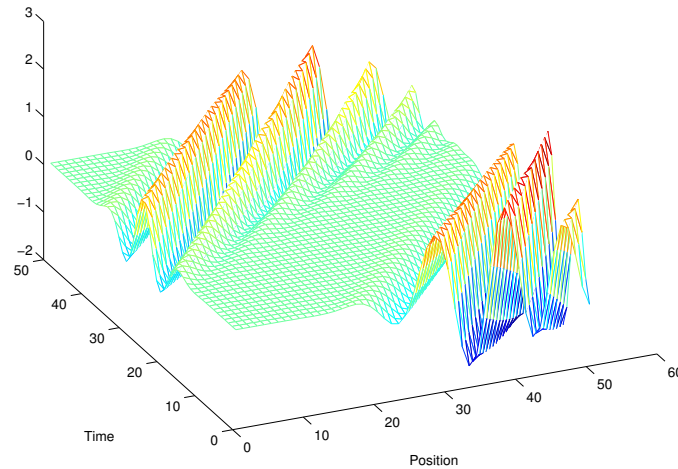
$$u(x, 0) = e^{-x^2/(2\sigma^2)} \cos(kx)$$

This initial profile must be preserved in shape by an accurate numerical code.

**FTCS fails to do this** for any choice of  $\tau$  and  $h$ .



Amplitude versus x and t



**Physical parameters:**  $\sigma = 0.1$ ,  $k = \pi/\sigma$ ,  $v = 1$

**FTCS parameters:**  $\tau = 0.002$ ,  $N = 500$ ,  $h = 0.02$ ,  $J = 50$

**Note:** Wave should have returned to its initial position with identical profile!

- The stability of a numerical method can be studied (before writing a programme !) by the following method.

### Fourier/Von Neumann Stability analysis

- Von Neumann stability analysis consists of inserting a trial solution into our scheme and seeing whether it will grow or not.

The trial solution is of the form,

$$u(x, t) = z(t)e^{ikx} \quad i = \sqrt{-1}$$

↙

growth factor

↘

spatial component which varies on length – scale  $k^{-1}$

This is just one Fourier component of the entire solution and on the grid this becomes

$$u_j^n = z_n e^{ik(jh)} = \xi^n e^{ik(jh)}$$

where  $\xi$  is the amplification factor;  $z_n = \xi z_{n-1}$ .

von Neumann stability analysis (which isn't very rigorous but always works) consists of inserting the trial solution and examining whether  $|\xi|$  (the amplification factor) lies between  $-1$  and  $+1$  or not.

- If we now insert this into the FTCS method

$$u_j^{n+1} = u_j^n - \frac{v\tau}{2h} (u_{j+1}^n - u_{j-1}^n)$$

Substitute in the trial solution:

$$\begin{aligned} z_{n+1} e^{ik(jh)} &= z_n e^{ik(jh)} - \frac{v\tau}{2h} \{ z_n e^{ik(j+1)h} - z_n e^{ik(j-1)h} \} \\ &= z_n e^{ik(jh)} \left\{ 1 - \frac{v\tau}{2h} [e^{ikh} - e^{-ikh}] \right\} \\ &= z_n e^{ik(jh)} \left\{ 1 - \frac{v\tau}{h} \cdot i \sin(kh) \right\} \end{aligned}$$

$\Rightarrow z_n = z_0 \xi^n$  where

$$\xi = 1 - \frac{v\tau}{h} \cdot i \sin(kh)$$

$$|\xi| = \sqrt{1 + \left(\frac{v\tau}{h}\right)^2 \sin^2(kh)}$$

$|\xi| > 1$  always (unless  $kh = n\pi$  which gives  $|\xi| = 1$ ).

- $|u_j^n|$  is amplified by the FTCS scheme and the method is unstable.
- The **Lax Equivalence Theorem**, which we will not prove, further states that the stability of a scheme is necessary and sufficient for convergence once the differencing scheme is consistent.

## Lax Method

- The simplest way to overcome the instability of the FTCS method is to replace the  $u_j^n$  term on the right hand side by

$$\frac{1}{2} (u_{j-1}^n + u_{j+1}^n)$$

$$u_j^{n+1} = \frac{1}{2} (u_{j+1}^n + u_{j-1}^n) - \frac{v\tau}{2h} (u_{j+1}^n - u_{j-1}^n) \quad (\text{Lax Method}) \quad (7)$$

This is the **Lax method** for the advection problem. As an exercise, adapt the file FTCS.m to solve the advection problem by the Lax method and show that

- For  $\tau \leq \frac{h}{v}$  the Lax scheme is stable. (We'll see shortly where this condition comes from).
  - But, for  $\tau \ll \frac{h}{v} = \tau_{\max}$  the pulse dies out!
- The reason for this “damping” is that the Lax method can be written as

$$\frac{u_j^{n+1} - u_j^n}{\tau} = -v \left( \frac{u_{j+1}^n - u_{j-1}^n}{2h} \right) + \frac{1}{2} \left( \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\tau} \right)$$

which is the FTCS representation of the equation

$$\frac{\partial u}{\partial t} = -v \frac{\partial u}{\partial x} + \frac{h^2}{2\tau} \frac{\partial^2 u}{\partial x^2}$$

“diffusion” term

- The Lax method has added **artificial viscosity** to the advection equation. If  $\tau \leq \frac{h}{v}$ , the artificial viscosity ensures that the method is stable (it smooths out the instabilities). But, if  $\tau$  is too small then the viscosity term dominates and pulse dies away (it smooths out everything!).
- Let us apply the Von Neumann stability analysis to the Lax method which gives

$$\begin{aligned} z_{n+1} e^{ik(jh)} &= \frac{1}{2} (z_n e^{ik(j+1)h} + z_n e^{ik(j-1)h}) - \frac{v\tau}{2h} (z_n e^{ik(j+1)h} - z_n e^{ik(j-1)h}) \\ &= z_n e^{ik(jh)} \left\{ \frac{1}{2} (e^{ikh} + e^{-ikh}) - \frac{v\tau}{2h} (e^{ikh} - e^{-ikh}) \right\} \end{aligned}$$

so that

$$\begin{aligned} \xi &= \cos(kh) - i \frac{v\tau}{h} \sin(kh) \\ \Rightarrow |\xi| &= \sqrt{\cos^2(kh) + \left( \frac{v\tau}{h} \right)^2 \sin^2(kh)} \end{aligned}$$

$$|\xi| \leq 1 \Leftrightarrow \left(\frac{v\tau}{h}\right)^2 \leq 1$$

**Conclusion:** The Lax method is numerically stable only if

$$\tau \leq \frac{h}{v}$$

This is known as the **Courant-Friedrich-Lewy** (CFL) stability criterion.

**Numerically** the value of  $u_j^{n+1}$  depends on  $u_{j-1}^n, u_j^n, u_{j+1}^n$

**Mathematically** in time  $\tau$  the solution just shifts to the right a distance  $v\tau$ .

$$\text{if } \tau > \frac{h}{v} \Leftrightarrow v\tau > h$$

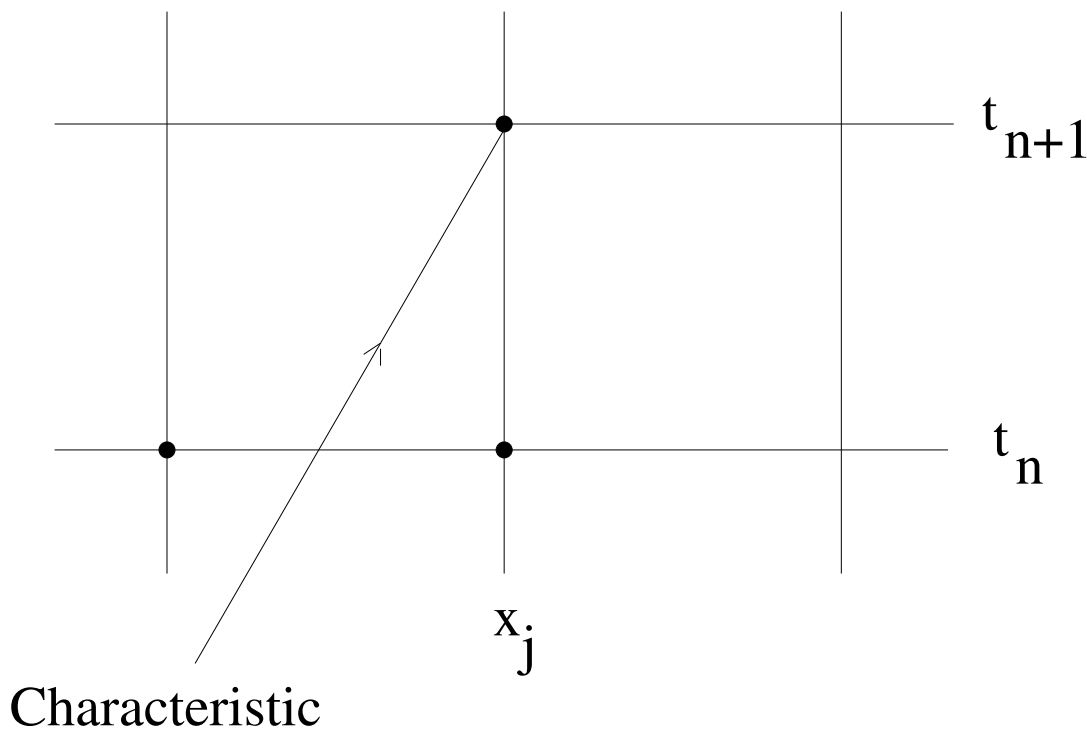
then  $u_j^{n+1}$  depends on the value of  $u^n$  at some point to the left of  $x_{j-1}$  which is not included in the numerical scheme



**Instability**

**Upwind scheme**

- Another way of obtaining a stable scheme for the advection equation is the **upwind scheme** which makes use of the concept of **characteristics**. Consider the simplest case where  $v(x, t) = v = \text{constant}$  (assume  $v > 0$ ) and draw a characteristic through the point  $(x_j, t_{n+1})$ .



- Forward time and backward space differencing gives (with  $\nu = v\tau/h$ )

$$\begin{aligned}\frac{u_j^{n+1} - u_j^n}{\tau} &= -v \frac{u_j^n - u_{j-1}^n}{h} \\ u_j^{n+1} &= (1 - \nu)u_j^n + \nu u_{j-1}^n\end{aligned}$$

Adapt FTCS.m to solve the advection problem by the upwind scheme

- Note that  $\nu = 1 \Rightarrow v = h/\tau$ , a characteristic goes through  $(x_{j-1}, t_n)$  and  $(x_j, t_{n+1})$  so that the upwind scheme gives the correct answer  $u_j^{n+1} = u_{j-1}^n$ . Similarly when  $\nu = 0$  we get  $u_j^{n+1} = u_j^n$ . For all cases where  $0 < \nu < 1$ , the upwind scheme gives a **linear interpolation** (see below) approximation to  $u_j^{n+1}$ .
- For a general  $v(x, t)$  this idea can be generalised to give the **upwind scheme** for the linear advection equation

$$u_j^{n+1} = \begin{cases} u_j^n - \nu(u_{j+1}^n - u_j^n) & \text{if } v_j^n < 0 \\ u_j^n - \nu(u_j^n - u_{j-1}^n) & \text{if } v_j^n > 0 \end{cases} \quad (8)$$

where  $\nu \equiv v_j^n \tau/h$ . We will write a programme for this method after the next section.

- Fourier analysis of the scheme's stability gives, when  $v_j^n > 0$ ,

$$\begin{aligned}\xi &= 1 - \nu(1 - e^{-ikh}) \\ \Rightarrow \xi^2 &= [(1 - \nu) + \nu \cos(kh)]^2 + [\nu \sin(kh)]^2 \\ &= (1 - \nu)^2 + \nu^2 + 2\nu(1 - \nu) \cos(kh) \\ &= 1 - 2\nu(1 - \nu)(1 - \cos(kh)) \\ \Rightarrow \xi^2 &= 1 - 4\nu(1 - \nu) \sin^2(kh/2)\end{aligned}$$

so that  $|\xi(k)| \leq 1$  for all  $k$  provided the **Courant Friedrich Lewy (CFL)** condition,  $\nu < 1$ , is satisfied.

- We also note the **Leapfrog scheme**

$$u_j^{n+1} = u_j^{n-1} - \nu(u_{j+1}^n - u_{j-1}^n) \quad (\text{Leapfrog method}) \quad (9)$$

which is centered time, centered space and is therefore not “self-starting” - we cant just use the initial condition to get it going. How could you get around this ? But it has the special property that if the CFL condition is satisfied then there is no damping, **prove that this is the case and adapt FTCS.m to use the leapfrog method.**

- We now study some of the above methods in practice with different initial conditions and advection speeds. The choices allowed are summarised in the following table. Note that the choice of  $v$  the exact solution.

$u(x, 0)$	$H(x - 0.1)H(0.3 - x)$ <b>shape1.m</b>	$\exp(-10(4x - 1)^2)$ <b>shape2.m</b>
$v(x, t)$	1 <b>adspeed1.m</b>	$(1 + x^2)/(1 + 2xt + 2x^2 + x^4)$ <b>adspeed2.m</b>
Exact	<b>exact1.m</b>	<b>exact2.m</b>

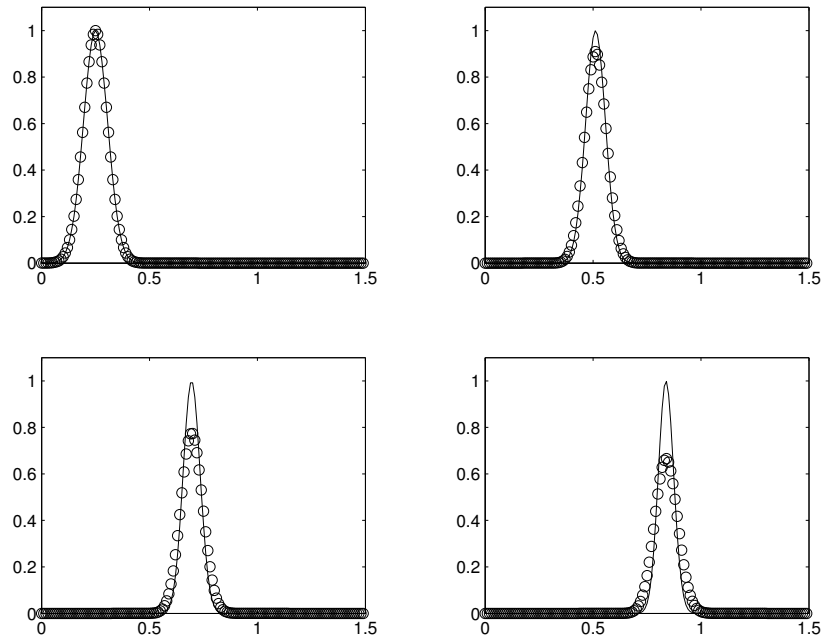
We are also free to choose either of the methods upwind (**upwind.m**) and Lax (**adlax.m**). The code can also run the Lax Wendroff method which is beyond the scope of these lectures but is discussed in the appendix for those who would like to learn more about slightly more advanced techniques.

N.B. These problems all have an analytic solution (solid curves on the plots) - we are testing the numerical methods and showing exactly how difficult it is to solve the advection problem on a computer.

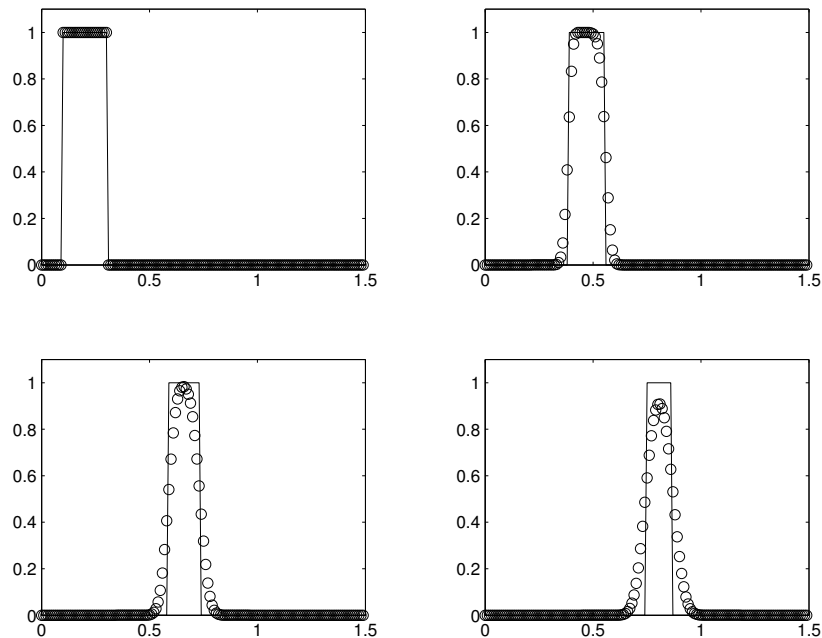
- Results are presented here for the non-uniform advection speed, with either initial condition and the upwind or Lax schemes; i.e. four different sets of results, and the analytic solution using characteristics is also plotted. In each case  $h = 0.1$  and the Courant number  $\equiv \tau v_{\max}/h = 1$ . Note that since  $v$  changes from point to point, on most of the x-axis we will therefore have a local Courant number which is less than 1. A full set of results should be generated by the student. The programme terminates before the wave profile runs off the grid.
- **N.B.** The programme calculates the timestep on the basis of fulfilling the CFL condition at the start of the programme. However,  $v(x, t)$  decreases with time (in the variable case) and this suggests that it might be a good idea to change the timestep after each iteration. Adapt the programme in order to this. Does this improve the results and why ?



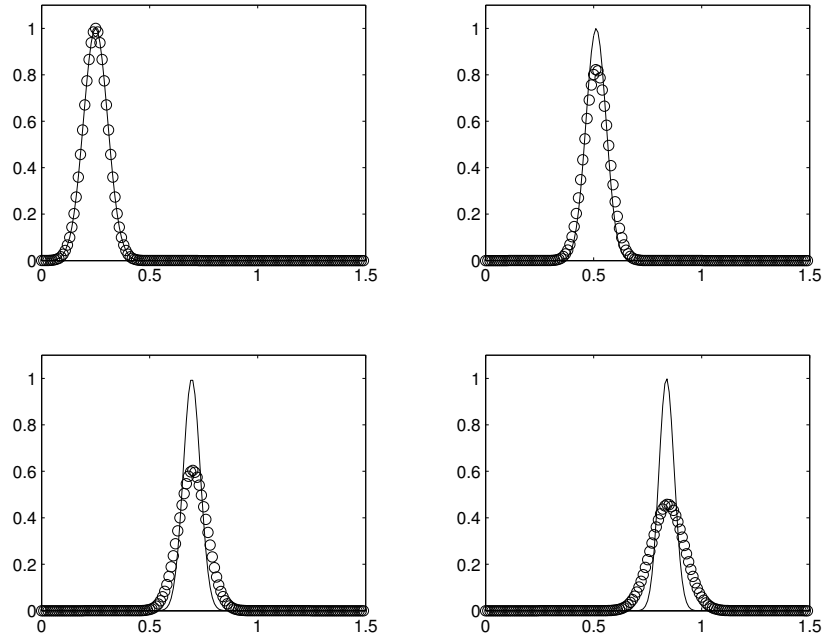
- The results for the **upwind method with an initial gaussian profile** are shown below for  $t = 0$ ,  $t = 1/3$ ,  $t = 2/3$  and  $t = 1$  and compared against the analytic solution (solid line); the same times apply to the subsequent three examples. It is clear that, although the peak of the numerical results moves with the correct speed, the upwind method reduces the height of the initial shape, i.e. an **amplitude error** which is quite large.



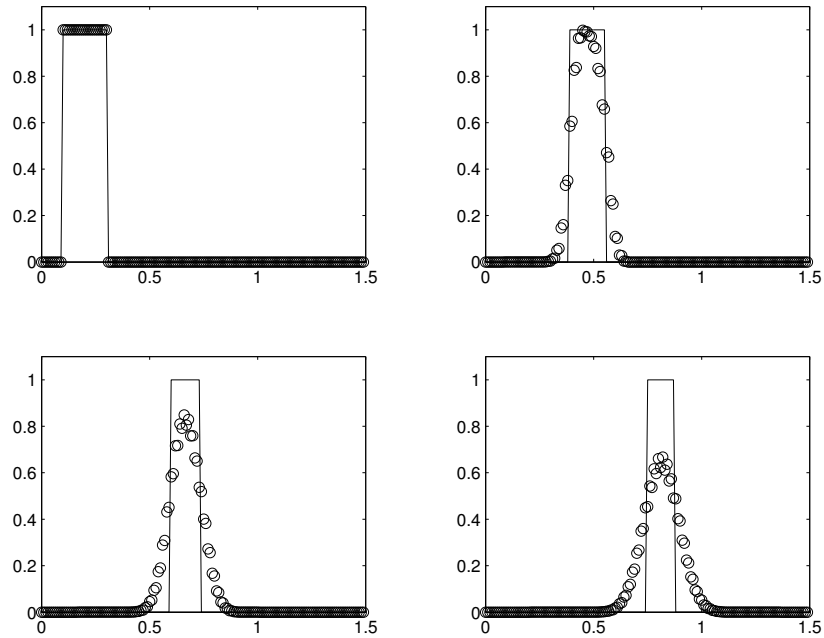
- The same qualitative features can be seen with the **upwind method for the top-hat initial condition**. Again the pulse moves at the correct speed but the amplitude is reduced and the edges of the pulse are smoothed out.



- The Lax scheme for the gaussian profile shows the damping effect one gets with a local courant number that is less than 1.



- The Lax method for the top-hat condition.



## Appendix

1. Consider the linear advection equation in the range  $0 < x < 1$  with the advection speed  $v(x) = x - 0.5$  and initial condition  $u(x, 0) = x(1 - x)$ . Sketch the initial condition and the characteristics. Why do we not need any extra conditions at the boundaries of  $x = 0$  and  $x = 1$ ? Solve for  $u(x, t)$ . What does the solution look like for  $t \rightarrow \infty$ ?
2. Consider the linear advection equation in the range  $0 < x < 1$  with the advection speed  $v(x) = 0.5 - x$  and initial condition  $u(x, 0) = x(1 - x)$ . Sketch the initial condition and the characteristics. Why do we need extra conditions at the boundaries of  $x = 0$  and  $x = 1$ ? With the boundary conditions  $u(0, t) = u(1, t) = 0$  solve for  $u(x, t)$ . What does the solution look like for  $t \rightarrow \infty$ ?
3. Solve each of the previous problems computationally and compare with the analytic results.
4. Show that when the CFL condition is satisfied by the leapfrog scheme for the linear advection scheme that there is no damping; i.e.  $|\xi| = 1$  if  $|v| < h/\tau$ . Write a programme which demonstrates this property.
5. For the linear advection equation  $u_t + vu_x$ , where  $v$  is a constant, the generalised upwind scheme is defined by

$$u_j^{n+1} = (1 - \theta)u_k^n + \theta u_{k-1}^n$$

where  $x_k - \theta h = x_j - v\tau$  and  $0 \leq \theta < 1$ . Apply Von Neumann stability analysis to this method and show that stability places no constraint on the choice of  $\tau$ .

## Further Reading

- There is a well known technique that is more accurate than anything we derived so far although its derivation is somewhat more complex. It is called the **Lax Wendroff** method and starts with a Taylor expansion of the analytic function

$$u(x, t + \tau) = u(x, t) + \tau u_t(x, t) + \frac{1}{2} \tau^2 u_{tt}(x, t) + O(\tau^3)$$

- The differential equation allows us to replace time derivatives by spatial derivatives which can then be central differenced.

$$u_t = -v u_x \Rightarrow u_{tt} = -v_t u_x - v u_{xt}$$

Noting that  $u_{tx} = -(v u_x)_x$  gives

$$u_{tt} = -v_t u_x + v (v u_x)_x$$

- The Taylor expansion can then be written as

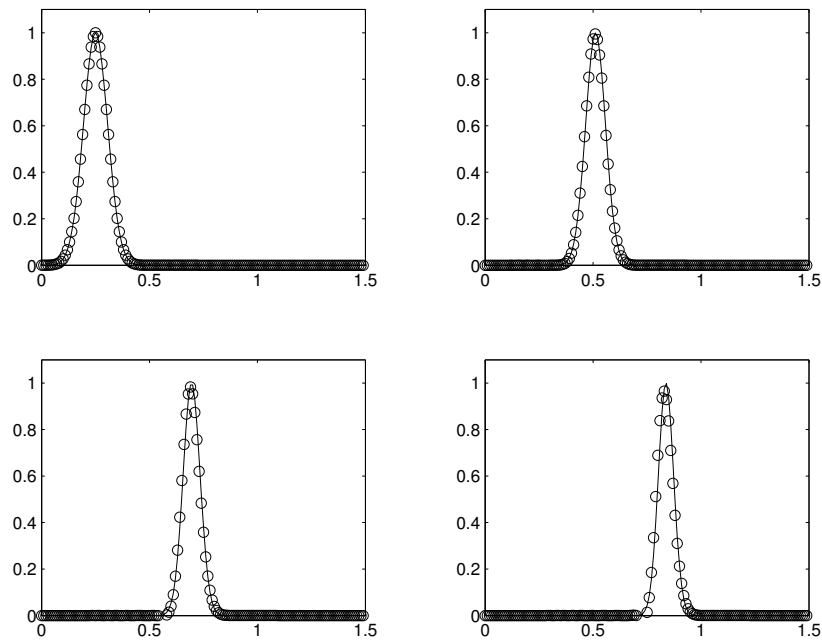
$$\begin{aligned} u(x, t + \tau) = u(x, t) & - v \tau u_x(x, t) \\ & + \frac{\tau^2}{2} [-v_t u_x(x, t) + v (v u_x)_x] \end{aligned}$$

Applying central differences for the spatial derivatives give the **Lax Wendroff scheme for the linear advection equation**

$$\begin{aligned} u_j^{n+1} &= u_j^n - \tau v_j^n \frac{u_{j+1}^n - u_{j-1}^n}{h} \\ &+ \frac{\tau^2}{2} \left[ - (v_t)_j^n \frac{u_{j+1}^n - u_{j-1}^n}{h} + v_j^n \frac{v_{j+1/2}^n (u_{j+1}^n - u_j^n) - v_{j-1/2}^n (u_j^n - u_{j-1}^n)}{h^2} \right] \end{aligned} \quad (\text{Lax Wendroff}) \quad (10)$$

so that it is therefore **second order accurate**.

- The **Lax Wendroff scheme for the gaussian profile** shows very good agreement between the numerical results and the analytic solution and represents a considerable improvement on the upwind method.



- However, **Lax Wendroff method for the top-hat condition** shows how difficult it is to reproduce non-smooth solutions; quite advanced schemes are needed !

