Chapter 3. Mathematics after Euclid

We have seen that the study of pure geometry was brought to a high degree of perfection by Euclid, who presented all the achievements of the previous three hundred years in his *Elements*. Following its foundation around 330 BCE, Alexandria became the centre of Greek intellectual life, and a famous library was established there c. 290 BCE. This library aimed to collect manuscript copies of all significant work across the whole field of human knowledge. One of its first librarians was the mathematician Eratosthenes, known for his so–called sieve, a method for isolating prime numbers.

From about the first century BCE, we can detect a gradual loss of originality and power in Greek mathematics. There certainly were major mathematicians following Euclid, working throughout the Greek cultural world. Foremost among these, we may mention Archimedes (c. 287–212 BCE), who was born in Syracuse in Sicily, and worked there for most of his life. He studied at Alexandria, probably with the successors of Euclid. On his return to Syracuse, he devoted himself to geometrical investigations, as well as to the design of mechanical contrivances. He is known particularly for the invention of a water screw. He also devised various war machines that were employed to protect Syracuse during a three year siege by the Romans. Historical accounts of his death at the hands of a Roman soldier during the capture of Syracuse have survived in the work of several authors. In 75 BCE, during a political appointment in Sicily, the Roman statesman and writer Cicero described finding, after much effort, the grave of Archimedes.

Foremost among his mathematical works that have survived, we mention:

- *On the sphere and cylinder.* This contains numerous propositions relating the dimensions of spheres, cones and cylinders. A representative example is *the surface of any sphere is equal to four times the greatest circle in it.* This means that the surface area of a sphere is $4\pi r^2$, where r is the radius of the sphere.

- *On the measurement of the circle.* Here he proves that the area of a circle is the same as that of a triangle whose base equals the circumference of the circle and whose height equals the radius of the circle. Proposition 2 is a proof of the inequality mentioned in Chapter 1, namely,

  $$3\frac{10}{71} < \pi < \frac{22}{7}.$$

  His method of approximation is to inscribe and circumscribe a circle by a regular polygon of 96 sides.

- *On conoids and spheroids.* This is a treatise of 40 propositions on surface areas and volumes of solids generated by the revolution of conic sections about their axes.
- *On spirals.* A book of 28 propositions relating to the curve now known as the spiral of Archimedes.
- *Equiponderants and centres of gravity.* This is an elementary work on mechanics.
- *On the quadrature of the parabola.* Here, he compares the area of a parabola with the area of a triangle. The method used anticipates the area summation of integral calculus, developed almost 2000 years after the his death.

For proving theorems about area and volume, Archimedes used the method of exhaustion introduced by Eudoxus and also the so–called *Lemma of Archimedes.* The lemma states:

> The excess by which the greater of two unequal areas exceeds the less can, if it be continually added to itself, be made to exceed any assigned finite area.

This principle, while it may seem reasonable to a modern mathematician, was not accepted by all Greek thinkers, and Aristotle rejected it. For this reason, Archimedes felt obliged to justify the use of the lemma in his preface to *On the quadrature of the parabola.*

The method of exhaustion involves a *reductio ad absurdum* and is in some respects similar to an argument in which one passes to a limit.

Archimedes's work was not well known in antiquity, although his name is mentioned by such later writers as Pappus, and it was only in the 6th century CE that commentaries on his work began to appear in Constantinople. In 1269, William of Moerbeke translated a fairly complete manuscript of Archimedes's work from Greek into Latin. By this means Archimedes became known in Western Europe and the translation was the source for all Renaissance copies of Archimedes's work. William was a monk of the Dominican order who travelled in Greece and Asia Minor. His translations from Aristotle, Archimedes and Proclus were especially influential in introducing Western European scholars to these neglected sources.

In the later years of the Hellenistic culture centred on Alexandria, much of the mathematical work that was undertaken consisted of commentaries, explanations and revisions of the major work done by Euclid and his contemporaries. There was little truly innovative work in mathematics from about 100 BCE onwards. Of course, there were exceptions to this statement, for we may mention such important figures as Diophantus and

Pappus, both of whom lived in Alexandria. Diophantus (3rd century CE?) is noteworthy as his major work, the *Arithmetica*, is devoted to problems of number theory and algebra, and appears to be an isolated achievement, without known predecessors or successors. It was not until the 16th century, when manuscript copies of the work of Diophantus eventually appeared in Italy, that Diophantine problems became known and appreciated by western European mathematicians. Translations of his work were to have great influence on number theorists in the 17th century, especially Fermat, whose famous last theorem concerns a problem in the Diophantine tradition. Pappus, of whom we know virtually nothing, is thought to have worked in the early fourth century CE. His treatise known as the *Collection*, containing varied results on geometry, has become important for the historical light it throws on earlier Greek mathematicians and their work.

While we have talked of the continuity of Greek mathematics being maintained through copying of original manuscripts and writing critical commentaries on the major works, it seems that by the fourth century CE, the library of Alexandria had all but ceased to exist. If this had survived, no doubt we would have much better understanding of the history of Greek mathematics and its practitioners, instead of having to rely on fragmentary and often unreliable anecdotal evidence. As it turns out, much of our knowledge of Greek mathematics arises from Arabic sources. Following the creation of Islam, in the early years of the 7th century CE, the Arabs conquered much of the territory previously held by the Byzantine Greeks. In 641, they captured Alexandria. The Arabs discovered Greek manuscripts dealing with mathematics and wished to know more of the subject. They made translations into Arabic of important Greek work, including Euclid's *Elements* and the *Arithmetica* of Diophantus. In some cases, the Arabic translations of these original Greek works are all that have survived, as the original Greek versions are in many cases lost. In this way, an Arabic tradition in geometry was established by the 8th century CE and it flourished throughout the Islamic world for several centuries.

Euclid's *Elements* played an important role in the Middle Ages. This was mainly due to the emphasis placed on logic in later medieval education. For several centuries following the collapse of the Roman empire, the *Elements* was known only by some fragments relating to elementary propositions in Books 1 and 2. These are believed to descend from translations made into Latin from the original Greek by the late Roman philosopher Boethius (died 524/525 CE). However, the significance of these fragments as part of a greater work was not appreciated.

In the 12th century, a surge of interest in scientific knowledge occurred in western Europe. Christian scholars travelled to Islamic Spain to study Arabic mathematical and scientific manuscripts. Translations of several key Greek works were made from Arabic into Latin, the most important being the translation of Euclid's *Elements*, by Athelhard of Bath, an English monk. Other translations followed but most versions of the *Elements* that circulated in western Europe until the 19th century were based on two or three translations made from Arabic into Latin in the 12th or 13th centuries, those of Gherard of Cremona and Johannes Campanus being the best known. Through these translations, the full extent of its 13 books became apparent to European scholars, instead of the much truncated version following from the Boethian tradition. Greek versions of the classical Greek texts were almost totally unknown in western Europe until the fall of Constantinople to the Turks in 1453, after which Greek manuscripts made their way into western European collections and libraries.

We turn now to a discussion of the gradual assimilation of the new number system that spread from the East during the Dark Ages of western Europe. Various alphabetic systems were used to denote numbers in ancient civilizations. We will concentrate on examining that used in ancient Greece. It seems that the Greeks used a decimal system of enumeration from the earliest times and the first known to us is the so–called Attic system. It employed I to denote the unit and five other symbols, namely Γ, Δ, H, X and M. These were the first letters of the words representing the numbers. Letters could be repeated up to four times to represent different numbers (as in the Roman system).

At some uncertain period of time, a new system that employed 27 small letters of the Greek alphabet was introduced. There was no symbol for zero, which the Greeks did not consider to be a number. The first nine letters represented the units 1 to 9, the next nine the tens from 10 to 90, and the third nine the hundreds from 100 to 900. To show that a numeral was intended, a horizontal stroke was placed above the letter in cursive writing. Thus,

$$\begin{aligned} \overline{\alpha} &= 1, \overline{\beta} = 2, \ldots, \overline{\theta} = 9 \\ \overline{\iota} &= 10, \ldots, \overline{\pi} = 80 \\ \overline{\rho} &= 100, \ldots, \end{aligned}$$

The Greek alphabet in normal use at the time consisted of 24 letters, and three extra letters, derived from the Phoenician alphabet and obsolete in literary use, were required to give the full 27 symbols. This system gave no indication of place value, unlike the Hindu–Arabic system, and historians generally think that it was a disadvantage to the development of arithmetic in Greek times.

By combining the letters, any number from 1 to 999 could be represented. For larger numbers, the same letters were employed with additional distinguishing marks adjoined.

Various devices, often very complicated, were used to represent fractions. We have explanations of Greek methods of elementary mathematical calculations, some of which have survived as papyrus fragments found in Egypt.

The Roman system used only the seven letters I, V, X, L, C, D and M. It is certainly true that such systems, while satisfactory for everyday life, were not well suited to abstract calculation, such as Archimedes used in his estimates of π.

One of the great advances in mathematics arose through the gradual adoption of the system of numbers that developed in India, over 1500 years ago. This system employed only the ten symbols 0, 1, 2, ..., 9 (or some form of these symbols), but its great advantage lay in its use of place value. This meant that the ten digits sufficed to represent any number in a way much more concise and readily understood compared with, say, the Roman way. Forms of Indian numerals can be traced back to the third century BCE. These evolved over the centuries into symbols which somewhat resemble the modern number symbols. There is however no evidence that the place value system was in use in India before the 6th century CE. Possibly through trading contacts, or by the acquisition of astronomical tables brought to Baghdad by an Indian ambassador in 773 CE, the Arabs became acquainted with the Indian (or Hindu, as it also called) number system. Information about this system was made available in an Arabic treatise by al–Khwarizmi, composed around 825 CE. No copies of the original work in Arabic have survived. It is thought to have been called *Kitab fi 'l–jam 'wa'l–tafriq* (Treatise on gathering and dispersion). It is now known only through reworkings of a Latin translation made in the 12th century. A 13th century manuscript of such a reworking in Cambridge University Library, although fragmentary, has been translated into English. From this, we can, for example, read about the rather lengthy way in which division was explained. Other more complete manuscripts on the topic have recently come to light (notably one in New York) and these give us a better understanding of al–Khwarizmi's work.

It appears that al–Khwarizmi's text explained the principles of place value, rules for addition and subtraction, doubling and halving, and multiplication and division. Medieval Europeans who attempted to explain the system followed the same plan. In Europe, the whole technique became known as *algorism* (a corruption of the name al–Khwarizmi).

The so–called Hindu–Arabic system of mumbers entered Europe mainly through Moslem Spain, although Greek manuscripts written in Constantinople in the 13th century show that the system was already known there and recognized as Indian. We do not know precisely when the Hindu–Arabic number system first appeared in northern Europe, but by studying manuscripts, scholars have been able to show that by the early 13th century, the Hindu–Arabic numerals were already used in English astronomical tables. Algorism in Europe was often taught through verse, the *Carmen de algorismo* (Song of algorism), written by a French monk in the 13th century, being a surviving example. In English it goes:

This present art is called 'algorismus' , in which

We make use of twice–five Indian figures:

$0 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

The first signifies one: two the second

The third signifies three: thus proceed left

Until you come to the end, which is called 'cifra'

Which signifies nothing: it gives significance to what is behind it

If you put any of these in the first place

It signifies simply itself: if in the second,

Itself tenfold, etc

The best known of the medieval algorisms is that of Johannes Sacrobosco (c1200–1250), known sometimes as John of Holywood. He may have been an Englishman, but few details of his early life are reliable. His *Algorismus* remained in use for over three centuries. Sacrobosco's treatise *Sphaera* on astronomy, based in part on the work of the Greek Ptolemy and his Arab commentators, was also widely studied until the 17th century.

The era known as the Dark Ages in western Europe began with the dissolution of the Roman empire at the end of the fifth century CE. This era is noteworthy for dominance of Christianity in the private and public domain, and for the comparative neglect of scientific and mathematical speculation. There was a feeling that God did not require humans to deal in experimentation or to seek to improve their worldly situation, as preparation for the afterlife was of far greater importance. Many of the scientific achievements of Greece and Rome were lost to western European culture for several centuries.

By the 12th century, the Dark Ages were coming to an end. Populations began to grow, as improvements occurred in agricultural methods. Commercial activity both in

Europe and further afield generated wealth. Contacts were made with the East, through the Crusades, through trade and by the exploratory visits of individual travellers. In Italy, several maritime city–states, including Amalfi, Genoa, Pisa and Venice, embarked on trading enterprises throughout the Mediterranean world, including Byzantium and the Moslem countries. Among these city–states was Pisa, whose population had risen to around 10,000 by the 12th century. Pisa was the birthplace of Leonardo, known as Fibonacci, who is considered to be the first significant mathematician of Christian western Europe. We do not know many details of Leonardo's personal life; 1170–1240 is thought to give a fairly accurate approximation to his life span. We do know about his mathematics, however, since five of his works have survived. As far as the history of the spread of the Hindu–Arabic number system is concerned, it is his first work *Liber abbaci* that is the most significant

In the prologue to *Liber abbaci*, Leonardo provides us with what little we know concerning his early life.

> After my father's appointment by his homeland as state official in the customshouse of Bugia for the Pisan merchants who thronged to it, he took charge; and in view of its future usefulness and convenience, had me in my boyhood come to him and there wanted me to devote myself to and be instructed in the study of calculation for some days. There, following my introduction, as a consequence of the marvellous instruction in the art, to the nine digits of the Hindus, the knowledge of the art very much appealed to me before all others, and for it I realized that all its aspects were studied in Egypt, Syria, Greece, Sicily, and Provence, with their varying methods; and at these places thereafter, while on business, I pursued my study in depth and learned the give–and –take of disputation. But all this even, and the algorism, as well as the art of Pythagoras I considered as almost a mistake in respect to the method of the Hindus. Therefore, embracing more stringently that method of the Hindus, and taking stricter pains in its study, while adding certain things from my own understanding and inserting also certain things from the niceties of Euclid's geometric art. I have striven to compose this book in its entirety as understandably as I could, dividing it into fifteen chapters. Almost everything which I have introduced I have diplayed with exact proof, in order that those seeking this knowledge, with its pre–eminent method, might be instructed and further, in order that the Latin people might not be discovered to be without it, as they have been up to now.

The town of Bugia, in modern Algeria, was a source of furs and leather, needed for two of Pisa's main industries. As Leonardo's brief prologue explains, he learned mathematics by studying with people proficient in the subject throughout the Mediterranean world. He became acquainted with Euclid's *Elements* and the algebra of al–Khwarizmi. He learned from Arab scientists the Hindu number system and its method of place value, and also the algorithms for arithmetic operations. On returning to Italy, he resolved to let the people of Italy know of the great utility of this number system by writing his treatise *Liber abbaci*, which first appeared in 1202. No copies of the original version have survived, but a new edition of 1228 exists in several copies. The work achieved considerable fame in its time, and a teaching curriculum based on it lasted for more than three centuries, but it was eventually forgotten until Baldassarre Boncompagni prepared an edition of the original Latin text in 1857. An English translation was made recently.

At the time when Leonardo wrote, calculations were usually made on an abacus (a calculating device that often used small stones or dust) and the answers were written down using Roman numerals. By contrast, the new system offered the opportunity both to calculate and express answers using the Hindu numerals, as we do today. This said, despite Leonardo's advocation of the new system, the old ways persisted in Italian commerce until at least the mid 15th century. Note incidentally that *Liber abbaci* does not translate as *Book of the Abacus*, despite appearances. *Abbaco* is the Italian name for calculating with the Hindu numerals and a *maestro d'abbaco* was a person skilled in the art of their use. Thus *Liber abbaci* means *Book of calculation by the Hindu method.*

The *Liber abbaci* is more than merely an introduction to the Hindu system of numbers. Indeed, it is an enormous work on a variety of mathematical topics, and its modern English version occupies over 600 printed pages! It treats much of what was known in the 12th century on arithmetic, algebra and problem solving. Some of the problems are rather contrived, and they are numerous, so the work becomes rather repetitive and a bit tedious to read.

Liber abbaci is divided into fifteen chapters, and reflects the original approach of al–Khwarizmi.

- On the recognition of the nine Indian figures and how all numbers are written with them; and how the numbers must be held in the hands, and on the introduction to calculations.

- On the multiplication of whole numbers.

- On the addition of them, one to the other.

- On the subtraction of lesser numbers from larger ones.

- On the multiplication of whole numbers with fractions and also fractions alone.

- On the addition, subtraction, and division of whole numbers with fractions and also the reduction of fractional parts into single parts.

- On the buying and selling of commercial and similar things.

- On the barter of commercial things and the buying of coin, and certain rules on the same.

- etc

- On the finding of square and cube roots, and the multiplication, division, or subtraction of them, and on the handling of binomials and apotomes and their roots.

- On the pertinent rules of geometric proportions; on problems of algebra and almuchabala.

We quote briefly from the first chapter, since it gives a description of the Indian number system.

> The nine figures are:
>
> 9 8 7 6 5 4 3 2 1
>
> With these nine figures, and the sign 0 which the Arabs call zephir any number whatsoever is written, as is demonstrated below. A number is a sum of units, . . . , and through the addition of them the numbers increase by steps without end. First, one composes from units those numbers which are from one to ten. Second, from the tens are made those numbers which are from ten up to one hundred, etc.

While the *Liber abbaci* was an innovative work, Leonardo's current fame is based on one small problem in the book, which has proved to be a source of endless further research. The problem occurs in Chapter 12, and is called *How many pairs of rabbits are created by one pair in one year.* Here it is:

> A certain man had one pair of rabbits together in a certain enclosed place, and one wishes to know how many are created from the pair in one year when it is the nature of them in a single month to bear another pair, and in the second month those to

> bear also. Because the abovewritten pair in the first month bore, you will double it; there will be two pairs in one month. One of these, namely the first, bears in the second month, and thus there are in the second month 3 pairs; of these in one month, two are pregnant, and in the third month 2 pairs of rabbits are born, and thus there are 5 pairs in the month; in this month 3 pairs are pregnant, and in the fourth month there are 8 pairs, of which 5 pairs bear another 5 pairs. *etc*

A modern version is along the following lines:

> Suppose that at the beginning of the year, a pair of rabbits is left in an enclosed place. These rabbits take two months to mature and after two months they produce another pair. Thereafter, they continually produce another pair each succeeding month. Each newborn pair takes two months to mature and after two months they also produce another pair each succeeding month. Assuming no deaths occur, find the number of rabbits on the island after n months.

Leonardo obtained the sequence

$$1, \quad 2, \quad 3, \quad 5, \quad 8, \quad 13, \quad 21, \quad 34, \quad 55, \quad 89, \quad 144, \quad 233, \quad 377,$$

so that at the end of the year there are 377 pairs of rabbits.

The modern approach to the problem is as follows. Let u_n denote the number of pairs after n months. Starting at $n = 0$, we have $u_0 = 1$. Similarly, $u_1 = 1$, but $u_2 = 2$ as the initial pair have produced a new pair after two months. Then $u_3 = 3$ but $u_4 = 5$, as both the original pair and the second pair have produced a new pair each. It is not too difficult to see that the next number in the sequence is found by adding its two immediate predecessors and thus we have the mathematical equation

$$u_{n+2} = u_{n+1} + u_n.$$

We call this a second order difference equation. The sequence of numbers obtained from Leonardo's problem is called the *Fibonacci sequence* (Leonardo was a member of the Bonacci family, and he styled himself *filio Bonacci*, which became Fibonacci–hence the other form of his name). Leonardo's sequence is defined by the initial values $u_1 = 1$ and $u_2 = 2$, but we can obtain different sequences by assigning different values to u_1 and u_2. Nonetheless, all such sequences have similar properties.

It is possible to write down an explicit formula for u_n, although it is a little complicated to use. We consider the roots α and β of the quadratic equation

$$x^2 = x + 1$$

associated to the difference equation $u_{n+2} = u_{n+1} + u_n$. We can take

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Then, by a standard method of solving difference equations, we obtain

$$u_n = \frac{\alpha^n}{\sqrt{5}} - \frac{\beta^n}{\sqrt{5}}.$$

As $|\beta| < 1$, the term $\beta^n \to 0$ as $n \to \infty$, and this means that, for large n, u_n is very accurately approximated by

$$\frac{\alpha^n}{\sqrt{5}}.$$

Furthermore, we can see that, as $n \to \infty$,

$$\frac{u_{n+1}}{u_n} \to \alpha = \frac{1+\sqrt{5}}{2} = 1.618033989\ldots$$

Thus the ratio of successive terms tends to a limit equal to α.

The roots α and β of the quadratic $x^2 = x + 1$ had already occurred in a celebrated problem of ancient Greek geometry. Suppose we have a line segment AB. A point C on this line segment, chosen so that AC is greater than CB, is said to divide the segment in *extreme and mean ratio* if

$$\frac{|AB|}{|AC|} = \frac{|AC|}{|CB|}.$$

Thus the ratio of the whole line segment to the larger segment equals the ratio of the larger segment to the smaller. Let $|AB| = a$ and $|AC| = b$. Then $|CB| = a - b$ and we obtain

$$\frac{a}{b} = \frac{b}{a-b}.$$

Hence $a^2 - ab = b^2$. Setting $r = a/b$, we see that $r^2 = r + 1$, and since we chose things so that $r > 1$, it follows that $r = \alpha$. We chose C so that $|AB| = \alpha|AC|$, so that $|AC|$ is $1/\alpha$ times the total length $|AB|$. Now $1/\alpha = \alpha - 1$ by properties of the quadratic, so that $|AC|$ is about 0.618 times the total length. The Greeks said that the line segment AB had been divided according to the *golden section.* This golden section, equal to 0.618 approximately or to

$$\frac{\sqrt{5}-1}{2}$$

precisely, is thought to have been especially appealing to the Greeks, and commentators have attempted to find occurrences of it in Greek architecture and art. Its significance for elegant design probably dates more accurately from the publication in 1509 of a book entitled *De divina proportione*, by Fra Luca Pacioli. This book is devoted to the study of the golden section and the five regular solids. It contains famous illustrations of geometrical figures, perhaps the work of Leonardo da Vinci.

The *Liber abbaci*, although it served a useful purpose in teaching the new system of numerals, and in encouraging the learning of problem–solving skills, did not exhibit great mathematical originality. Leonardo, however, published four other mathematical works that are known to us. These are:

- *Practica geometriae* (1220)
- *Flos* (1225)
- Letter to the philosopher Theodorus
- *Liber quadratorum* (1225)

In these publications, he was able to show his skill in more theoretical mathematics, and it is clear that he had learnt much from ancient sources, such as Euclid and Diophantus. We can sum up Leonardo's contributions to the mathematics of the Middle Ages by noting that his works were copied for several centuries, and he was frequently quoted with approval by later mathematicians such as Pacioli and Cardano. His exposition of the new methods of computation had an immediate influence, and he is considered to be the teacher of the masters of calculation in Italy (*maestri d'abbaco*). The more theoretical parts of his algebra and number theory were mainly ignored until a greater interest in mathematics for its own sake developed a few centuries later. Leonardo is nonetheless considered to be the first great mathematician of Christian Western Europe.