CHAPTER 8: DIFFERENTIAL CALCULUS

1. Rules of Differentiation

As we have seen, calculating derivatives from first principles can be laborious and difficult even for some relatively simple functions. It is clearly not a very feasible approach if we want to find the derivative of a more complicated function such as

$$f(x) = \frac{\sqrt[3]{x} + 11x^4}{2\sqrt{x} - 4x^7}.$$

However, the power of calculus as a tool lies in the fact that functions such as this *can* be differentiated without great effort and (after some practice) with almost no thought. Just as for limits, there are rules for calculating the derivatives of more complicated function which are constructed from simple ingredients (such as power functions). It is the use of these rules to differentiate functions which is called the *Differential Calculus.*

1.1. The Power Rule for Differentiation.

We have already calculated (from first principles) the derivatives of several power functions.

Example 1.1. The function $f(x) = x^0$ is just the constant function $f(x) = 1$. Thus,

$$\frac{d}{dx}(x^0) = 0.$$

Example 1.2. The function $f(x) = x^1$ is just the identity function $f(x) = x$. Thus,

$$\frac{d}{dx}(x^1) = 1.$$

Example 1.3. We have calculated the derivative of $f(x) = x^2$:

$$\frac{d}{dx}(x^2) = 2x.$$

Example 1.4. The functions $f(x) = 1/x$ is the power function $f(x) = x^{-1}$. Thus

$$\frac{d}{dx}(x^{-1}) = -x^{-2}.$$

Example 1.5. The function $f(x) = \sqrt{x}$ is the power function $f(x) = x^{1/2}$. Thus,

$$\frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2}.$$

Observe that, in each case, when we differentiate a power of x, the resulting power is the original power reduced by 1, and that this new power of x is *multiplied by* the original exponent.
This is the general rule which works for *every* function:

$$\text{For any } r \in \mathbb{R}, \quad \frac{d}{dx}x^r = rx^{r-1}.$$

i.e., if $f(x) = x^r$, then $f'(x) = rx^{r-1}$.
Of course, there is nothing privileged about the letter x here:

$$\begin{aligned} \frac{d}{dt}t^r &= rt^{r-1} \\ \frac{d}{ds}s^r &= rs^{r-1} \\ \frac{d}{dw}w^r &= rw^{r-1} \\ \frac{d}{d\eta}\eta^r &= r\eta^{r-1} \\ \frac{d}{d\text{Tom}}\text{Tom}^r &= r\text{Tom}^{r-1} \\ \ldots\text{etc} \end{aligned}$$

Example 1.6.

$$\frac{d}{dx}(x^3) = 3x^2.$$

Example 1.7.

$$\frac{d}{dx}(x^7) = 7x^6.$$

Example 1.8. This general power rule is consistent with the calculations that we have already made for special power functions, even in the case $r = 0$ and $r = 1$:

$$\begin{aligned} \frac{d}{dx}(x^0) = 0x^{-1} = 0 \\ \frac{d}{dx^1} = 1 \cdot x^0 = 1 \cdot 1 = 1 \end{aligned}$$

Example 1.9.

$$\frac{d}{dx}x^\pi = \pi x^{\pi-1}$$

Some care needs to be taken with fractional and negative powers.

Example 1.10.

$$\frac{d}{dx}(1/x^4) = \frac{d}{dx}(x^{-4}) = -4x^{-5} = \frac{-4}{x^5}$$

Example 1.11.

$$\begin{aligned}\frac{d}{dx}(1/\sqrt[3]{x}) &= \frac{d}{dx}(x^{-1/3}) \\ &= -\frac{1}{3}x^{-4/3} \\ &= \frac{-1}{3\sqrt[3]{x^4}}\end{aligned}$$

Example 1.12. Calculate the derivative of the function $f(x) = \sqrt{x^3}/\sqrt[3]{x^2}$.
Solution: From the properties of exponents we have

$$\frac{\sqrt{x^3}}{\sqrt[3]{x^2}} = \frac{x^{3/2}}{x^{2/3}} = x^{\frac{3}{2}-\frac{2}{3}} = x^{5/6}.$$

Thus

$$f'(x) = \frac{d}{dx}x^{5/6} = \frac{5}{6}x^{-1/6} = \frac{5}{6\sqrt[6]{x}}.$$

1.2. **The Sum Rule / Difference Rule for Differentiation.** If we know how to differentiate two functions $f(x)$ and $g(x)$, the sum rule tells us how to differentiate the sum of the two functions.

$$\boxed{\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)}$$

Example 1.13.

$$\begin{aligned}\frac{d}{dx}(x^3 + \sqrt{x}) &= \frac{d}{dx}(x^3) + \frac{d}{dx}(\sqrt{x}) \\ &= 3x^2 + \frac{1}{2\sqrt{x}}.\end{aligned}$$

Example 1.14.

$$\frac{d}{dx}(\sqrt{x} - x^5 + \frac{1}{x}) = \frac{1}{2\sqrt{x}} - 5x^4 - \frac{1}{x^2}.$$

Example 1.15.

$$\frac{d}{dx}(3 + x^2 + \sqrt[3]{x}) = 0 + 2x + \frac{1}{3}x^{-2/3} = 2x + \frac{1}{3\sqrt[3]{x^2}}.$$

1.3. **The Constant Rule for Differentiation.**

$$\boxed{\frac{d}{dx}C \cdot f(x) = C \cdot \frac{d}{dx}f(x) \qquad (C \text{ a constant})}$$

Example 1.16.

$$\frac{d}{dx}(10\sqrt[5]{x}) = 10 \cdot \frac{d}{dx}\sqrt[5]{x} = 10 \cdot \frac{1}{5}x^{-4/5}.$$

Example 1.17.

$$\frac{d}{dx}(15x^7) = 15 \cdot 7x^6 = 105x^6.$$

We can combine the constant and sum/difference rules:

Example 1.18.

$$\frac{d}{dx}(5x^3 - 6x^2 + 4x + 12) = 5 \cdot 3x^2 - 6 \cdot 2x^2 + 4 \cdot 1 + 0 = 15x^2 - 12x + 4$$

The sum rule for derivatives resembles the sum rule for limits. This is because the former is a straightforward consequence of the latter:

Proof of the sum rule:

Let $s(x) = f(x) + g(x)$

Then $\frac{d}{dx}s(x) = s'(x)$

$$\begin{aligned}
&= \lim_{h\to 0} \frac{s(x+h) - s(x)}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} \\
&= \lim_{h\to 0} \left\{ \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right\} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&\quad \text{(sum rule for limits)} \\
&= f'(x) + g'(x)
\end{aligned}$$

1.4. **The Product Rule for Differentiation.**

If u and v are functions of x then

$$\boxed{\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}}$$

i.e., $(uv)'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x)$

Example 1.19.

$$\begin{aligned}
\frac{d}{dx}(\underbrace{x^5}_{u}\underbrace{(\sqrt{x} + x^3)}_{v}) &= x^5 \frac{d}{dx}(\sqrt{x} + x^3) + (\sqrt{x} + x^3)\frac{d}{dx}x^5 \\
&= x^5(\frac{1}{2\sqrt{x}} + 3x^2) + (\sqrt{x} + x^3) \cdot 5x^4
\end{aligned}$$

[Of course, we could also proceed as follows: $x^5(\sqrt{x} + x^3) = x^5(x^{1/2} + x^3) = x^5x^{1/2} + x^5x^3 = x^{11/2} + x^8$ and now use the sum and power rules. We will see more interesting uses of the product rule when we learn to differentiate some more interesting functions: trigonometric functions, exponentials and logarithms.]

Example 1.20. Suppose that we have a function $f(x)$ and that all we know about it is its value and its rate of growth at $x = 3$. Suppose, for example, that we know that $f(3) = -15$ and $f'(3) = 11$. Let $g(x)$ be the function $g(x) = x^2 f(x)$. What is $g'(3)$?
Solution: By the product rule,

$$g'(x) = x^2 f'(x) + 2x f(x).$$

Thus

$$g'(3) = 9f'(3) + 6f(3) = 9 \cdot 11 + 6 \cdot (-15) = 99 - 90 = 9.$$

[The product rule for limits says that the limit of a product is just the product of the two limits. Why, then, is the product rule for differentiation such a complicated rule?
Proof of the Product Rule:
Let $p(x) = u(x) \cdot v(x)$

$$\tfrac{d}{dx}(uv) = p'(x) = \lim_{h\to 0} \tfrac{p(x+h)-p(x)}{h}$$

$$= \lim_{h\to 0} \tfrac{u(x+h)\cdot v(x+h) - u(x)\cdot v(x)}{h}$$

$$= \lim_{h\to 0} \tfrac{u(x+h)v(x+h) - u(x)v(x+h) + u(x)v(x+h) - u(x)v(x)}{h}$$

$$= \lim_{h\to 0} \tfrac{[u(x+h)-u(x)]\cdot v(x+h) + u(x)\cdot[v(x+h)-v(x)]}{h}$$

$$= \lim_{h\to 0} \tfrac{u(x+h)-u(x)}{h} \cdot \lim_{h\to 0} v(x+h) + u(x) \cdot \lim_{h\to 0} \tfrac{v(x+h)-v(x)}{h}$$

$$= u'(x) \cdot v(x) + u(x) \cdot v'(x).]$$

1.5. **The Quotient Rule for Differentiation.** If u and v are two functions of x then

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

Equivalently, letting $u = f(x)$, $v = g(x)$, we can write this rule in the form:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

Caution: Because of the *minus sign* in the formula, the order in which the terms appear is very significant. The formula can be expressed as

$$\frac{\text{bottom} \cdot \text{derivative of top} - \text{top} \cdot \text{derivative of bottom}}{(\text{bottom})^2}.$$

Example 1.21.

$$\begin{aligned}\frac{d}{dx}\left(\frac{3x+2}{x^3+x}\right) &= \frac{(x^3+x)\frac{d}{dx}(3x+2)-(3x+2)\frac{d}{dx}(x^3+x)}{(x^3+x)^2}\\ &= \frac{(x^3+x)\cdot 3-(3x+2)(3x^2+1)}{(x^3+x)^2}\\ &= \frac{(3x^3+3x)-(9x^3+3x+6x^2+2)}{(x^3+x)^2}\\ &= \frac{-6x^3-6x^2-2}{(x^3+x)^2}.\end{aligned}$$

Example 1.22. Find the equation of the tangent line to the graph of the function

$$f(x)=\frac{x^3+1}{x^2+x-1}$$

at $x=1$ ($y=2$).

Solution: We first find the derivative of $f(x)$ at an arbitrary point x:

$$\begin{aligned}f'(x)=\frac{d}{dx}\left(\frac{x^3+1}{x^2+x-1}\right) &= \frac{(x^2+x-1)\frac{d}{dx}(x^3+1)-(x^3+1)\frac{d}{dx}(x^2+x-1)}{(x^2+x-1)^2}\\ &= \frac{(x^2+x-1)3x^2-(x^3+1)(2x+1)}{(x^2+x-1)^2}\\ &= \frac{(3x^4+3x^3-3x^2)-(2x^4+x^3+2x+1)}{(x^2+x-1)^2}\\ &= \frac{x^4+2x^3-3x^2-2x-1}{(x^2+x-1)^2}\end{aligned}$$

Thus

$$f'(1)=\frac{1+2-3-2-1}{1^2}=-3.$$

This is the *slope* of the tangent line. The tangent line contains the point $(1,2)$. Therefore, the equation of the line is $y-2=-3(x-1)=-3x+3$. So $y=-3x+5$ is the line in question.