CHAPTER 23: INVERSE TRIGONOMETRIC FUNCTIONS

1. The Inverse of the sin function

The sine function is neither strictly increasing nor strictly decreasing on $\mathbb{R}$. Furthermore, it takes the value 1 (or any other number in $[-1, 1]$) at infinitely many different inputs. Thus, it certainly does not have an inverse, *unless we restrict its domain.*

To obtain a function which has an inverse, we restrict the domain of the sine function to an interval on which it is *strictly increasing.* The Inverse Function Theorem then guarantees us the existence of an inverse function. Thus,

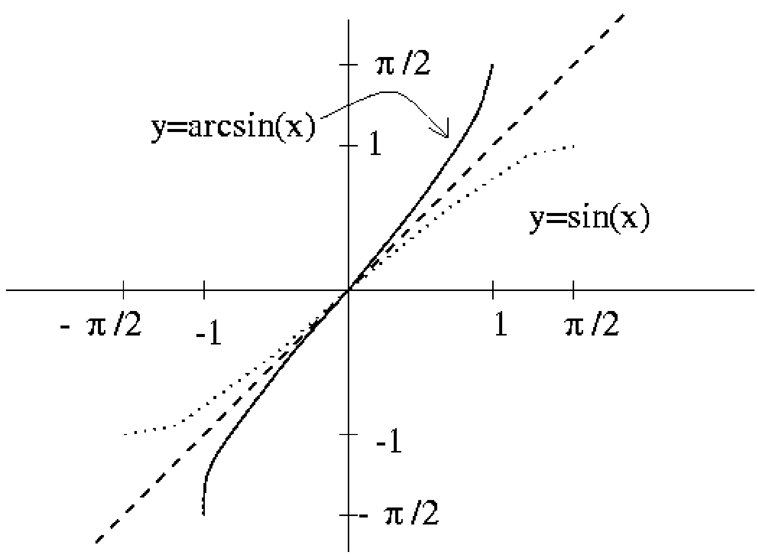
$$\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$$

is continuous and strictly increasing.

Thus, by the Inverse Function Theorem, there is an inverse function

$$\sin^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$$

Using the principle that the graph of the inverse of a function is just the mirror image of the graph of the function through the line $y = x$, we can graph this function:



($\sin^{-1}$ is also called 'arcsin' in some texts).

[**Caution:** *Don't* confuse $\sin^{-1} x$ with $\frac{1}{\sin x} = \csc x$!]

1.1. **Some values of** $\sin^{-1}$**.** Recall that if $\sin(a) = b$ then $\sin^{-1}(b) = a$. Thus:

$$\begin{aligned} \sin(0) = 0 &\implies \sin^{-1}(0) = 0 \\ \sin(\pi/2) = 1 &\implies \sin^{-1}(1) = \pi/2 \\ & \quad\;\; \sin^{-1}(-1) = -\pi/2 \\ & \quad\;\; \sin^{-1}(1/2) = \pi/6 \\ & \quad\;\; \sin^{-1}(1/\sqrt{2}) = \pi/4 \\ & \quad\;\; \ldots \text{and so on} \end{aligned}$$

$\sin^{-1}(x)$ is differentiable on $(-1, 1)$ by the Inverse Function Theorem.
To find a formula for the inverse, we proceed as for the exponential function.

$$\begin{aligned} y = \sin^{-1}(x) &\iff \sin(y) = x \\ \frac{d}{dx}\sin y &= \frac{d}{dx}x \\ \cos(y)\cdot\frac{dy}{dx} &= 1 \\ \implies \frac{dy}{dx} &= \frac{1}{\cos y} \end{aligned}$$

We would like to express this in terms of x. Thus, we wish to express $\cos y$ in terms of $x = \sin y$.
Now

$$\begin{aligned} \cos^2 y &= 1 - \sin^2 y \\ \implies \cos y &= \sqrt{1 - \sin^2 y} \\ &= \sqrt{1 - x^2} \end{aligned}$$

So

$$\boxed{\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \quad x \in (-1, 1)}$$

Hence

$$\boxed{\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C}$$

1.2. **Some Related integrals.**

$$\begin{aligned}
\int \frac{1}{\sqrt{9-x^2}}dx &= \\
& \frac{1}{3}\int \frac{1}{\sqrt{1-(x/3)^2}}dx \\
(\text{Let } u = x/3 &, \quad du = \frac{1}{3}dx) \\
&= \int \frac{1}{\sqrt{1-u^2}}du \\
&= \sin^{-1} u + C \\
&= \sin^{-1}(x/3) + C
\end{aligned}$$

Similarly

$$\boxed{\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + C}$$

Example 1.1. Find

$$\int \frac{1}{\sqrt{7-x^2}}\ dx$$

Solution: Here $a = \sqrt{7}$, so the integral equals

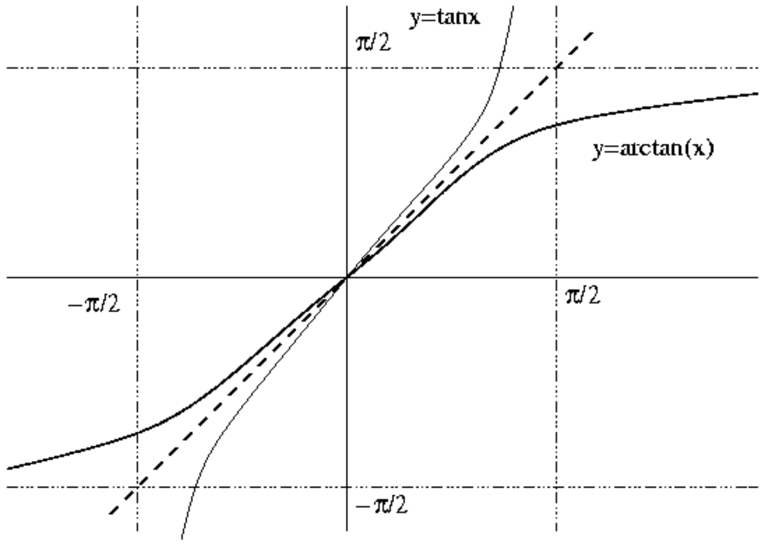
$$\sin^{-1}\left(\frac{x}{\sqrt{7}}\right) + C$$

2. The inverse of $\tan x$

Consider the tan function, but with its domain restricted to the interval $(-\pi/2, \pi/2)$.
$\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is differentiable and strictly increasing.
So, by the Inverse Function Theorem again, there is a differentiable inverse

$$\tan^{-1} = \arctan : \mathbb{R} \to (-\pi/2, \pi/2)$$

It's graph looks like:



Some values of $\tan^{-1}$:

$$\begin{aligned} \tan(\pi/4) = 1 \quad \Longrightarrow \quad & \tan^{-1}(1) = \pi/4 \\ & \tan^{-1}(0) = 0 \\ & \tan^{-1}(-1) = -\pi/4 \\ & \ldots \text{ etc} \end{aligned}$$

The lines $x = \pm\pi/2$ are vertical asymptotes of $\tan x$. Thus, the lines $y = \pm\pi/2$ are *horizontal* asymptotes of $y = \tan^{-1}(x)$ (this means, of course, that $\lim_{x\to\infty} \tan^{-1} x = \pi/2$).
Next, we differentiate $\tan^{-1}(x)$:

$$\begin{aligned} y &= \tan^{-1} x \\ \tan y &= x \\ \sec^2 y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\sec^2 y} \end{aligned}$$

We must express $\sec^2 y$ in terms of $x = \tan y$. Now

$$\begin{aligned} \sec^2 y &= \frac{1}{\cos^2 y} \\ &= \frac{\cos^2 y + \sin^2 y}{\cos^2 y} \\ &= 1 + \frac{sin^2 y}{\cos^2 y} \\ &= 1 + \tan^2 y \end{aligned}$$

Thus

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{1+\tan^2 y} \\ &= \frac{1}{1+x^2} \end{aligned}$$

$$\boxed{\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}}$$

Thus

$$\boxed{\int \frac{1}{1+x^2} dx = \tan^{-1} x + C}$$

2.1. **Some Related integrals.**

$$\begin{aligned} \int \frac{1}{x^2+4}\, dx &= ? \\ &= \frac{1}{4}\int \frac{1}{(x/2)^2+1}\, dx \\ \text{Let } u = \frac{x}{2},\ du &= \frac{1}{2}dx,\ \frac{1}{2}du = \frac{1}{4}dx \\ &= \frac{1}{2}\int \frac{1}{u^2+1} du \\ &= \frac{1}{2}\tan^{-1} u + C \\ &= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C \end{aligned}$$

In general

$$\boxed{\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C}$$

Example 2.1. Find

$$\int \frac{1}{x^2+5}\, dx$$

Solution: Here, $a = \sqrt{5}$. The answer is

$$\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$$