

CHAPTER 14: OPTIMISATION PROBLEMS

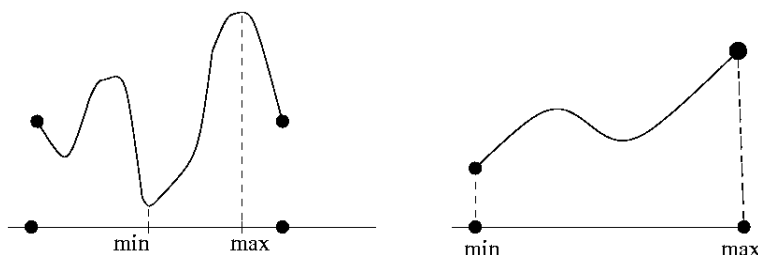
1. MAXIMA AND MINIMA OF FUNCTIONS

A (global) maximum of a function is a point *in the domain* at which the function achieves its highest value:

Definition 1.1. Suppose $f : D \rightarrow \mathbb{R}$ is any real-valued function. $a \in D$ is said to be a (global) *maximum* (resp. (global) *minimum*) if

$$\begin{aligned} f(a) &\geq f(x) \quad \text{for every } x \in D \\ (\text{resp. } f(a) &\leq f(x) \quad \text{for every } x \in D) \end{aligned}$$

Typical pictures:



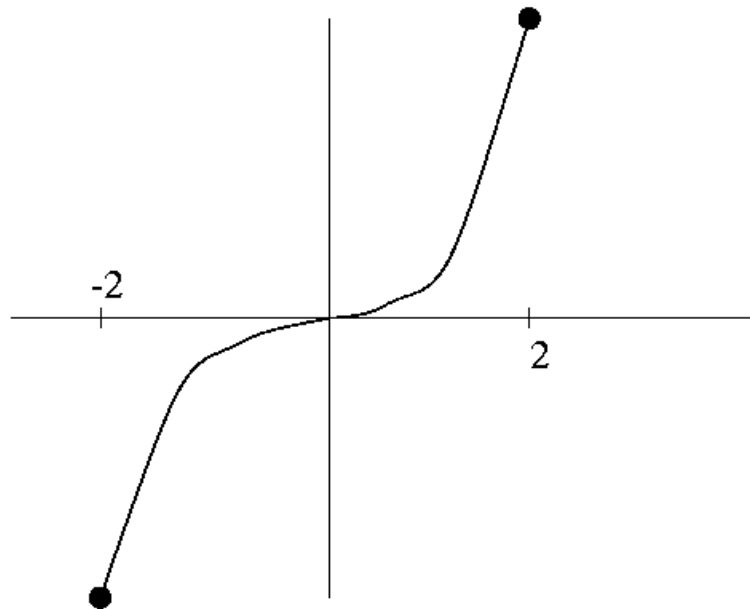
Note: If $a \in D$ is a maximum (there may be more than one – see below), then $f(a)$ is *the maximum value* of f .

Example 1.1. Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2 + 1$. 0 is the minimum. 1 is the minimum value. There is no maximum.

Example 1.2. Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^4 - 2x^2 + 1 = (x^2 - 1)^2$. This function takes the value 0 at -1 and at 1 . It never takes negative values (why?). Thus -1 and 1 are (global) minima of this function. 0 is the minimum value.

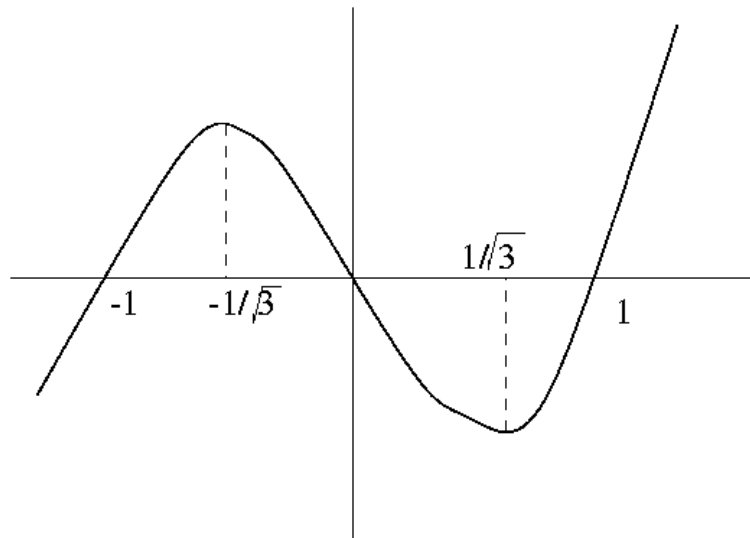
Example 1.3. Consider $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3$. This has no maximum or minimum.

Example 1.4. Consider, however, the function $f : [-2, 2] \rightarrow \mathbb{R}$, $f(x) = x^3$:



-2 is the minimum and 2 is the maximum. 8 is the maximum value.

Example 1.5. Consider $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3 - x$:



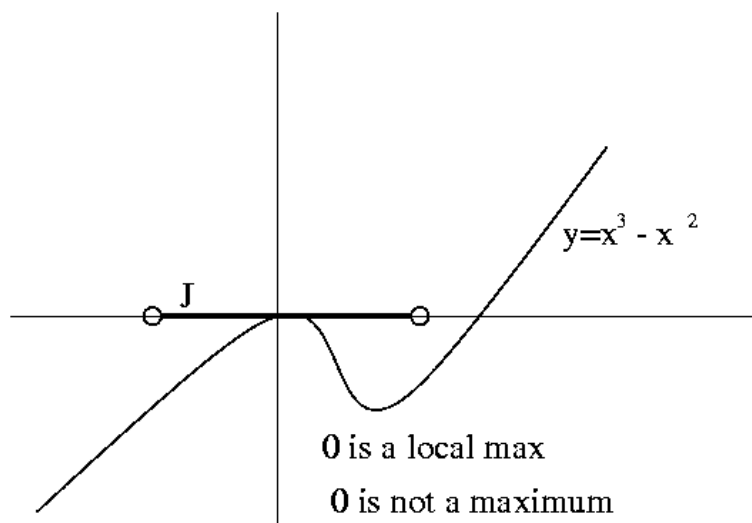
This function has no maximum or minimum.

$-1/\sqrt{3}$ is a *local* maximum. $1/\sqrt{3}$ is a *local* minimum.

Here is the precise definition of the term *local maximum*:

Definition 1.2. c is said to be a *local maximum* of f (respectively *local minimum* of f) if c belongs to some *open* interval J contained in the domain of f for which it is true that $f(c) \geq f(x)$ for every $x \in J$ (respectively, $f(c) \leq f(x)$ for every $x \in J$).

Example 1.6. Consider $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3 - x^2$:



Example 1.7. Consider the function $f(x) = 1 - x^2$.
0 is a local maximum, and a (global) maximum.

Example 1.8. Consider the function $f : [-2, 2] \rightarrow \mathbb{R}$, $f(x) = x^3$. (See example 1.4 above.) 2 is a global maximum, but is *not* a local maximum; there is no open interval J containing 2 which is contained in the domain of f .

We are now in a position to provide a proof of the First Derivative Theorem:

Theorem 1.1 (First Derivative Theorem). *If c is a local maximum or minimum of f and if $f'(c)$ exists, then $f'(c) = 0$.*

Proof: Suppose that c is a local maximum. So $f(x) \leq f(c)$ if x is near c (i.e for x in some open interval J containing c). Thus $f(c+h) \leq f(c)$ if h is small. So $f(c+h) - f(c) \leq 0$ if h is small. Thus if h is small and *positive*

$$\frac{f(c+h) - f(c)}{h} \leq 0$$

So

$$f'(c) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} \leq 0 \quad (*)$$

But if h is small and *negative*

$$\frac{f(c+h) - f(c)}{h} \geq 0$$

$$\Rightarrow f'(c) \geq 0 \quad (**)$$

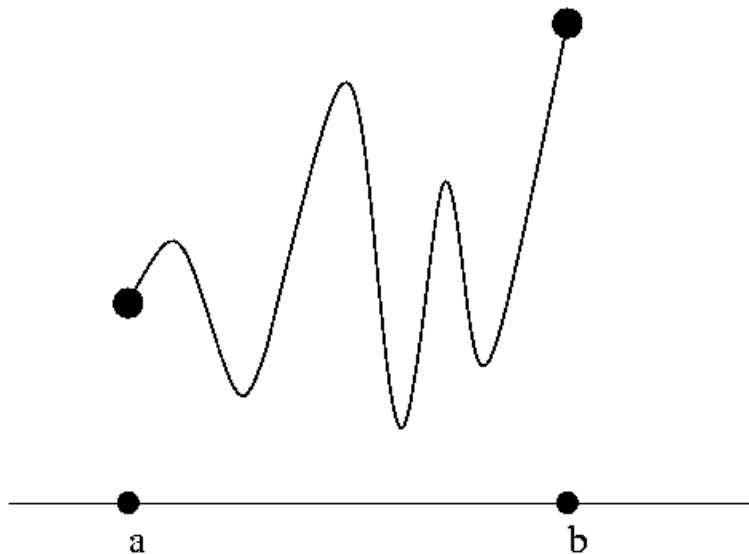
(*) and (**) imply $f'(c) = 0$.

[A similar argument works if c is a local minimum. You should verify this for yourself.]

Caution: As we've seen, $f'(c) = 0$ does *not* imply that c is a local maximum or minimum in general.

Theorem 1.2 (Min-Max Theorem for Continuous Functions). *If $f : [a, b] \rightarrow \mathbb{R}$ is continuous on the finite closed interval $[a, b]$, then f has a global maximum and minimum in $[a, b]$.*

Proof: Extremely difficult.



Example 1.9. The function $f : (-1, 1) \rightarrow \mathbb{R}$, $f(x) = x/(x^2 - 1)$ is continuous on the finite interval $(-1, 1)$. It has no maximum or minimum in the interval $(-1, 1)$. (It has vertical asymptotes at -1 and 1 and is strictly decreasing on $(-1, 1)$ – sketch its graph.) Of course, the interval $(-1, 1)$ is not *closed*.

Note: For a continuous function on $[a, b]$, the maximum and minimum will occur *either* at the critical points *or* at the endpoints. You must *always* check the endpoints.

Example 1.10. Find the maximum and minimum of $f(x) = 2x^3 - 9x^2 + 12x$ on $[0, 3]$.

Solution: $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2)$. So the critical points are 1 and 2. Now $f(1) = 5$ and $f(2) = 4$. However, we must also check the endpoints: $f(0) = 0$ and $f(3) = 9$. So 0 is the minimum and 3 is the maximum. (It is easily seen that 1 is a local maximum and 2 is a local minimum, however).

2. OPTIMIZATION; APPLICATIONS OF MAXIMA AND MINIMA

Example 2.1. A fish farmer is stocking a lake with fish. The more fish the greater the competition for food, and so the fish gain weight more slowly. She knows from experiments that with N fish per unit area the average amount of weight each fish gains in a season is

$$w = 600 - 30N \text{ g}$$

Which value of N will lead to the maximum *total* production of weight of fish?

Solution: We need to express this as a mathematical problem about the maximum or minimum of a function on a given domain.

By reading the last sentence of the question, we see that the function is ‘total production of weight’ and that it is a function of N , the number of fish per unit area. Now

Let T = total production of weight

The information in the question allow us to express T in terms of N :

$$\begin{aligned} T &= \text{number of fish} \times \text{wt. gain per fish} \\ &= N \cdot (600 - 30N) \\ &= 600N - 30N^2 \end{aligned}$$

We want the global maximum of the function T for $N \geq 0$.

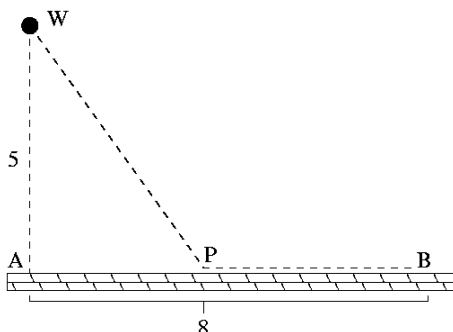
Note that T is a *quadratic*, so we know that it has a unique critical point, which is a *maximum* since the leading coefficient $-30 < 0$.

$$\frac{dT}{dN} = 600 - 60N$$

Thus $N = 10$ is the critical point.

Conclusion: Maximum weight production occurs by introducing 10 fish per unit area.

Example 2.2. An offshore oil well is located in the ocean (at point W), 5 km from the closest point, A, on the shore. The oil is to be piped to a point B on the shore which is 8 km from A, by piping it under water to a point P on the shore and then from P to B along the shoreline. The cost of laying pipe underwater is 100,000 euro per km and on land is 75,000 euro per km. Where should P be located so that the cost is minimized?



Solution: Let C denote the total cost.

Let x = distance from A to P. So $8 - x$ is the distance from P to B.

By Pythagoras’s theorem, the distance from W to P is

$$\sqrt{5^2 + x^2} = \sqrt{25 + x^2}$$

Thus the total cost is

$$C = 100\sqrt{25 + x^2} + 75 \cdot (8 - x) \quad (\text{in 1000 euro})$$

where $0 \leq x \leq 8$.

Thus we want the minimum of C on $[0, 8]$.

This is a continuous function on a finite closed interval. We need only check the values of the function at the critical points and the endpoints:

Critical points:

$$\begin{aligned} \frac{dC}{dx} &= 100 \cdot \frac{1}{2\sqrt{25 + x^2}} \cdot 2x - 75 \\ &= \frac{100x}{\sqrt{25 + x^2}} - 75 \end{aligned}$$

Solve

$$\begin{aligned} \frac{100x}{\sqrt{25 + x^2}} &= 75 : \\ 4x &= 3(\sqrt{25 + x^2}) \\ 16x^2 &= 9 \cdot (25 + x^2) = 225 + 9x^2 \\ 7x^2 &= 225 \\ x^2 &= \frac{225}{7} \\ x &= \sqrt{\frac{225}{7}} = \frac{15}{\sqrt{7}} \end{aligned}$$

Endpoints: 0, 8

$$\begin{aligned} C(0) &= 100 \cdot 5 + 75 \cdot 8 = 1100 \\ C(8) &= 100 \cdot \sqrt{25 + 64} = 100\sqrt{89} \approx 943.4 \\ C\left(\frac{15}{\sqrt{7}}\right) &= 100 \left(\sqrt{25 + \frac{225}{7}} \right) + 75 \left(8 - \frac{15}{\sqrt{7}} \right) \approx 930.7 \end{aligned}$$

Conclusion: Cost is minimised when distance from A to P is $15/\sqrt{7} \approx 5.67$ km.

Example 2.3. The concentration, K , of a medicine in the bloodstream t hours after it is taken is

$$K = \frac{0.03t}{1 + t^2} \quad \text{for } t \geq 0$$

At what time t is the concentration greatest? Sketch the graph of K against t .

Solution: Note that K is a *rational function* of t . Since $1 + t^2$ has no roots, it has no vertical asymptotes. However, the t -axis is a horizontal asymp.

(i.e., as $t \rightarrow \infty$, $K \rightarrow 0$).

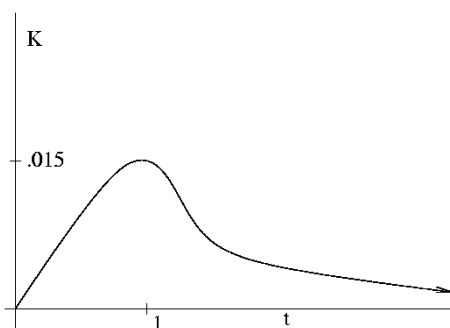
$$\begin{aligned} \frac{dK}{dt} &= \frac{(1+t^2) \cdot (.03) - (.03t) \cdot 2t}{()^2} \\ &= (.03) \cdot \frac{1+t^2-2t^2}{()^2} \\ &= (.03) \frac{1-t^2}{(1+t^2)^2} \end{aligned}$$

Critical points: $t = 1$ (and $t = -1$)

$$\begin{array}{c|c} \hline K'(0) & K'(2) \\ \hline = .03(1/1) > 0 & = (.03)(-3)/()^2 < 0 \\ \swarrow & \searrow \\ \end{array}$$

Thus $t = 1$ is a local (and global) maximum (on the domain $[0, \infty)$).

At $t = 1$, $K = .015$; at $t = 0$, $K = 0$.

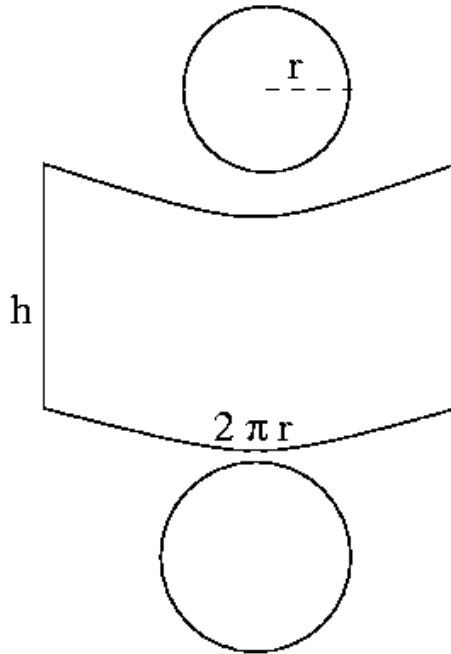


Conclusion: The concentration level increases from 0 to a maximum of .015 after 1 hour and thereafter slowly decreases.

Example 2.4. A closed cylindrical can is to hold 1 litre ($= 1000\text{cm}^3$) of oil. What height and radius should it have in order to minimize the amount of material needed to manufacture the can?

Solution: Let $r =$ radius, $h =$ height, $A =$ amount of material.

We begin by expressing A as a function of both r and h :



$$A = 2\pi r^2 + 2\pi r h$$

However, we are told that the *volume* of the can is 1000; i.e. $\pi r^2 h = 1000$. Use this to eliminate one of our variables:

$$h = \frac{1000}{\pi r^2}$$

So

$$\begin{aligned} A &= 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2} \\ &= 2\pi r^2 + \frac{2000}{r} \end{aligned}$$

We want to minimize A on the domain $(0, \infty)$ (where $r > 0$)

A minimum, if it exists, will be a critical point. Observe that A is a rational function of r .

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2} = 0$$

$$\pi r = \frac{500}{r^2}$$

$$r^3 = \frac{500}{\pi}$$

$$r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42\text{cm}$$

$A'(1) = 4\pi - 2000 < 0$	$A'(10) = 40\pi - 20 > 0$
↘	↗

The table shows that A is decreasing before the critical point and increasing afterwards. Thus the critical point is indeed a global minimum. When

$$\begin{aligned} r &= \sqrt[3]{\frac{500}{\pi}} \\ h &= \frac{1000}{\pi r^2} = \frac{1000}{\pi(500/\pi)^{2/3}} \\ &= 2 \cdot \left(\frac{500}{\pi}\right)^{1/3} = 2r \end{aligned}$$

Conclusion: The amount of material used is minimized when $r = \sqrt[3]{500/\pi} \approx 5.42\text{cm}$ and $h = 2r$