CHAPTER 18: ANTI-DIFFERENTIATION

1. A Fundamental Preliminary Result

The problem which is addressed in this chapter is the problem of calculating all functions which have some given derivative. In order to do that we first have to determine exactly when two different functions have the same derivative.

If two functions differ by a constant then they have the same derivative (since the derivative of a constant is 0).

The converse statement is also true; *if two functions have the same derivative then they must differ by a constant.*

We state this more precisely as a theorem:

Theorem 1.1. *Suppose the $F(x)$ and $G(x)$ are continuous on the interval $[a, b]$ and differentiable on its interior and that*

$$G'(x) = F'(x) \qquad \textit{for all } x \in (a, b)$$

Then $G(x) = F(x) + C$ for some constant C.

(The proof of this fact is difficult.)

2. Antiderivatives: Indefinite Integration

Often we have information about the derivative of an unknown function and we use this to find out about the function itself.

Example 2.1. A body is dropped from a height. What is its *velocity* after t seconds, given that the *acceleration* due to gravity is 9.8 $\mathrm{m/s^2}$.

Solution: We are looking for a formula for the velocity $v = v(t)$. What information do we have?

We know that acceleration $= \dfrac{dv}{dt} = -9.8\ \mathrm{m/s^2}$.

Thus v is a function whose derivative is -9.8.

What are the possibilities?:

Here are some:

$$\begin{aligned} v &= -9.8t \\ v &= -9.8t + 1 \\ v &= -9.8t + C \end{aligned}$$

Are there any other possibilities that we haven't conceived of?

No: For if we let $F(t) = -9.8t$ then $v'(t) = -9.8 = F'(t)$ for all t and so $v(t) = F(t) + C$ by Theorem 1.1; i.e. $v(t) = -9.8t + C$ for some constant C.

What is C in this particular example?
At time $t = 0$, $v = 0$ so that

$$0 = v(0) = -9.8 \cdot 0 + C = C \Longrightarrow C = 0$$

Conclusion: $v = -9.8t$.

In this example we know that the derivative of v is -9.8. We say that v is an *antiderivative* of -9.8.

Definition 2.1. In general, if $F'(x) = f(x)$ we say that $F(x)$ is *an antiderivative of* $f(x)$.

Suppose that we have found *an* antiderivative, $F(x)$, of a given function, $f(x)$. So $f(x)$ is the derivative of $F(x)$.
If $G(x)$ is any other antiderivative of $f(x)$, then $G(x)$ and $F(x)$ have the same derivative (namely, $f(x)$) and hence $G(x) = F(x) + C$ (by Theorem 1.1).
Thus $F(x) + C$ is the *general antiderivative of* $f(x)$ (in the sense that any function of this form is an antiderivative of $f(x)$ and any antiderivative of $f(x)$ is of this form). We use the following notation to express this fact:

$$\int f(x)dx = F(x) + C$$

The left-hand side in this equation is called the *indefinite integral of* $f(x)$ *with respect to* x. The function $f(x)$ in this formula is called *the integrand.* This notation is due to Leibniz. Its origin will be explained later.

Example 2.2. Find $\int x\, dx$. i.e., find all functions whose derivative is x.
Solution: Below we will see that there is a simple rule (The Power Rule) for solving this problem. For the present we will use trial and error.
We wish to find a function whose derivative is x. Every time we differentiate a power of x, the power decreases by 1. So our first guess is x^2:

$$\frac{d}{dx}(x^2) = 2x$$

This is not quite what we want, but it is only out by a factor of 2, so

$$\frac{d}{dx}\left(\frac{x^2}{2}\right) = x$$

i.e., $F(x) = x^2/2$ is an antiderivative of x.
Thus, by the remarks above, the general antiderivative is $x^2/2 + C$. We write

$$\int x dx = \frac{x^2}{2} + C$$

Note: The statement

$$\int f(x)dx = F(x) + C$$

is *entirely equivalent to* the statement

$$\frac{d}{dx}F(x) = f(x)$$

Thus any time we have a differentiation formula (eg. $\frac{d}{dx}\sin x = \cos x$), we can immediately write down a corresponding integration formula ($\int \cos x\, dx = \sin x + C$).

For instance, we derive a power rule for integration from the power rule for differentiation:

Theorem 2.1 (The Power Rule for Integration)**.** *If* $n \neq -1$ *then*

$$\boxed{\int x^n dx = \frac{x^{n+1}}{n+1} + C}$$

Proof: We must prove that the derivative of the right-hand side is the integrand, x^n:

$$\begin{aligned} \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) &= \frac{1}{n+1}\frac{d}{dx}x^{n+1} \\ &= \frac{1}{n+1}\cdot(n+1)x^n = x^n \end{aligned}$$

Example 2.3.

$$\int x^3 dx = \frac{x^4}{4} + C$$

Example 2.4.

$$\int x^{87} dx = \frac{x^{88}}{88} + C$$

Example 2.5.

$$\begin{aligned} \int \frac{1}{x^2} dx = \int x^{-2} dx &= \frac{x^{-1}}{-1} + C \\ &= -\frac{1}{x} + C \end{aligned}$$

Example 2.6.

$$\begin{aligned} \int \sqrt{x} dx = \int x^{1/2} dx &= \frac{x^{3/2}}{3/2} + C \\ &= \frac{2}{3}x^{3/2} + C \end{aligned}$$

Example 2.7.

$$\int \frac{1}{x} dx = ?$$

The power rule for integration does not cover this case. The correct answer here is a function we have not yet met, and which will be dealt with later.

3. RULES FOR INTEGRATION

3.1. The Sum/Difference Rule for Integration.

If $\int f(x)dx = F(x) + C$ and if $\int g(x)dx = G(x) + C$ then $\int f(x) \pm g(x)dx = F(x) \pm G(x) + C$;i.e.,

$$\boxed{\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx}$$

Proof: This is derived immediately from the corresponding differentiation rule:

$$\begin{aligned} \frac{d}{dx}\left(F(x) \pm G(x)\right) &= \frac{d}{dx}F(x) \pm \frac{d}{dx}G(x) \\ &= f(x) \pm g(x) \end{aligned}$$

Example 3.1.

$$\int x^3 + \sqrt{x}\, dx = \int x^3\, dx + \int \sqrt{x}\, dx = \frac{x^4}{4} + \frac{2}{3}x^{3/2} + C.$$

Example 3.2.

$$\int x^{43} - x^{42}\, dx = \frac{x^{44}}{44} - \frac{x^{43}}{43} + C.$$

3.2. The Constant Rule for Integration.

$$\boxed{\int af(x)dx = a\int f(x)dx \quad (a \text{ constant})}$$

Example 3.3.

$$\int 5x^6\, dx = 5\int x^6\, dx = 5 \cdot \frac{x^7}{7} + C.$$

We can, of course, combine the sum, power and constant rules.

Example 3.4.

$$\begin{aligned} &\int 3x^2 + 5x^3 dx = \\ &3 \cdot \frac{x^3}{3} + 5 \cdot \frac{x^4}{4} + C \\ &= x^3 + \frac{5}{4}x^4 + C \end{aligned}$$

Example 3.5.

$$\begin{aligned} &\int 2x^3 - 5\sqrt{x}\ dx \\ &= 2 \cdot \frac{x^4}{4} - 5 \cdot \frac{2}{3}x^{3/2} + C \\ &= \frac{1}{2}x^4 - \frac{10}{3}x^{3/2} + Cx \end{aligned}$$

Example 3.6.

$$\begin{aligned}
&\int \frac{3x^2+5x^3+1}{x^2}dx \\
&= \int 3+5x+\frac{1}{x^2}dx \\
&= 3x+\frac{5}{2}x^2-\frac{1}{x}+C
\end{aligned}$$

NOTE: *There is no product or quotient rule for integration.*

4. Trigonometric Functions

$$\frac{d}{dx}\sin x = \cos x \Longrightarrow$$

$$\boxed{\int \cos x dx = \sin x + C}$$

$$\frac{d}{dx}\cos x = -\sin x \Longrightarrow$$

$$\boxed{\int \sin x dx = -\cos x + C}$$

Thus, for example,

$$\int 3\cos x + 2\sin x dx = 3\sin x - 2\cos x + C$$

We still have much unfinished business:

$$\int \tan x dx = ? \qquad \textstyle\int \sec x dx = ?$$

$$\frac{d}{dx}2^x = ? \qquad \textstyle\int 2^x dx = ?$$

$$\frac{d}{dx}\log x = ?$$

We will deal with all of these questions later. It turns out that the key to answering all of these questions is to find an antiderivative of the function

$$f(x) = \frac{1}{x}.$$

In order to do this, we will use a method of constructing antiderivatives based on the concept of *area*. How are **areas** related to **antiderivatives**? This is the topic of the next section.