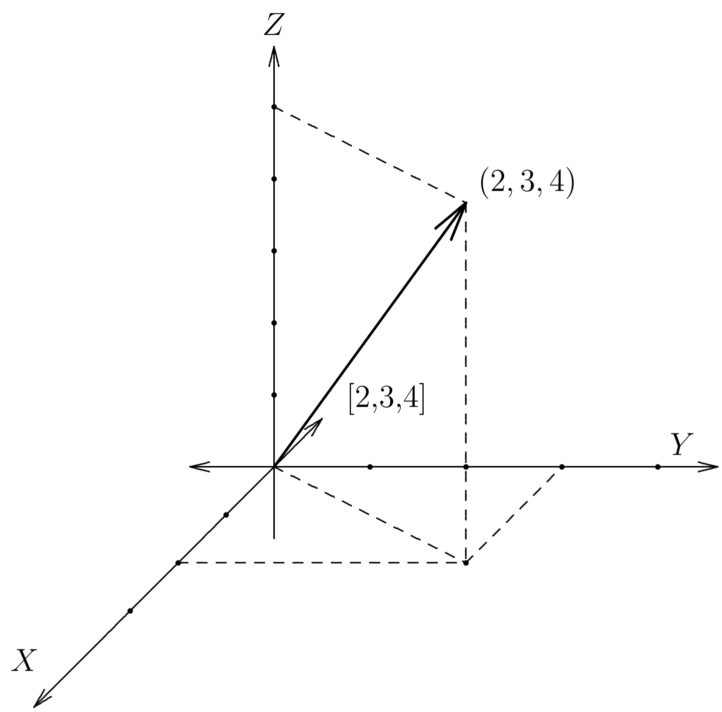
Chapter 5

Vector Geometry in $\mathbb{R}^3$

5.1 Vectors in $\mathbb{R}^3$

3-dimensional Euclidean space, denoted $\mathbb{R}^3$, is described by three orthogonal coordinate axes labelled X, Y and Z as shown. A point in $\mathbb{R}^3$ is described by by an *ordered triple* of real numbers, comprising its X, Y and Z coordinates.



As with $\mathbb{R}^2$, a vector in $\mathbb{R}^3$ is a directed line segment in $\mathbb{R}^3$, and may be described by its components; i.e. the coordinates of its terminal point if its initial point is located at the origin O $(0,0,0)$.

All of our results on vectors in $\mathbb{R}^2$ extend to $\mathbb{R}^3$ in a natural way. We list some of these.

1. Components If $A = (1,-2,3)$ and $B = (-2,2,1)$ are points in $\mathbb{R}^3$, the components of the vector $\vec{AB}$ are given by

   "coordinates of B"-"coordinates of A"

   $$\vec{AB} = [-2-1, 2-(-2), 1-3] = [-3,4,2]$$

2. Length If $\vec{v} = [1,2,-2]$ is a vector in $\mathbb{R}^3$, then

   $$||\vec{v}|| = \sqrt{(1)^2 + (2)^2 + (-2)^2} = \sqrt{9} = 3$$

   In general if $\vec{v} = [a,b,c]$ then $||\vec{v}|| = \sqrt{a^2+b^2+c^2}$.

3. Scalar Product If $\vec{u} = [1,3,-2]$ and $\vec{v} = [2,-1,0]$ then

   $$\vec{u}.\vec{v} = 1(2) + 3(-1) + (-2)(0) = -1$$

   In general

   $$[a_1,b_1,c_1].[a_2,b_2,c_2] = a_1a_2 + b_1b_2 + c_1c_2$$

   As in $\mathbb{R}^2$, $\vec{u}.\vec{v} = ||\vec{u}||\,||\vec{v}||\cos\theta$ for any two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ and angle θ between them. In particular $\vec{u} \perp \vec{v}$ if and only if $\vec{u}.\vec{v} = 0$, for non-zero vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$.

4. Projections Again as in $\mathbb{R}^2$: for vectors $\vec{u}$, $\vec{v}$ in $\mathbb{R}^3$

   $$\text{proj}_{\vec{u}}\vec{v} = \frac{\vec{u}.\vec{v}}{||\vec{u}||^2}\vec{u}; \qquad \text{perp}_{\vec{u}}\vec{v} = \vec{v} - \text{proj}_{\vec{u}}\vec{v} = \vec{v} - \frac{\vec{u}.\vec{v}}{||\vec{u}||^2}\vec{u}$$