3.3 The Inverse of a 3×3 Matrix

For any $n \times n$ matrix the adjoint and determinant are defined and satisfy

$$A \times \mathrm{adj}(A) = \mathrm{adj}(A) \times A = \det(A) \times I_n$$

Provided $\det(A) \neq 0$, A is invertible and

$$A^{-1} = \frac{1}{\det(A)} \times \mathrm{adj}(A)$$

(If $\det(A) = 0$, A does not have an inverse).

However the definition of the determinant and adjoint are much more complicated than in the 2×2 case.

Example 3.3.1 *Let* $A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & -2 & 1 \\ -4 & 1 & -1 \end{pmatrix}$. *Find* A^{-1}

The Adjoint of A

Step 1 *The Matrix of Minors*

For each entry $(A)_{ij}$ of A, we define the *minor* M_{ij} of $(A)_{ij}$ to be the determinant of the 2×2 matrix which remains when the ith row and jth column (i.e. the row and column containing $(A)_{ij}$) are deleted from A.

We begin by computing the 9 minors :

$$M_{11} : M_{11} = \det \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} = -2(-1) - (1)(1) = 1$$

$$M_{12} : M_{12} = \det \begin{pmatrix} 2 & 1 \\ -4 & -1 \end{pmatrix} = 2(-1) - (1)(-4) = 2$$

$$M_{13} : M_{13} = \det \begin{pmatrix} 2 & -2 \\ -4 & 1 \end{pmatrix} = 2(1) - (-2)(-4) = -6$$

$$M_{21} : M_{21} = \det \begin{pmatrix} 3 & 0 \\ 1 & -1 \end{pmatrix} = 3(-1) - (0)(1) = -3$$

$$M_{22} : M_{22} = \det \begin{pmatrix} 1 & 0 \\ -4 & -1 \end{pmatrix} = 1(-1) - (0)(-4) = -1$$

$$M_{23} : M_{23} = \det\begin{pmatrix} 1 & 3 \\ -4 & 1 \end{pmatrix} = 1(1) - (3)(-4) = 13$$

$$M_{31} : M_{31} = \det\begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix} = 3(1) - (0)(-2) = 3$$

$$M_{32} : M_{32} = \det\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = 1(1) - (0)(2) = 1$$

$$M_{33} : M_{11} = \det\begin{pmatrix} 1 & 3 \\ 2 & -2 \end{pmatrix} = 1(-2) - (3)(2) = -8$$

We now write the *matrix of minors* M of A defined by

$$(M)_{ij} = \text{the minor of } (A)_{ij}.$$

$$M = \begin{pmatrix} 1 & 2 & -6 \\ -3 & -1 & 13 \\ 3 & 1 & -8 \end{pmatrix}$$

Step 2 *The Matrix of Cofactors* We define the *cofactor* C_{ij} of the entry $(A)_{ij}$ of A as follows:

$$\begin{aligned} C_{ij} &= M_{ij} \quad \text{if} \quad i+j \text{ is even} \\ C_{ij} &= -M_{ij} \quad \text{if} \quad i+j \text{ is odd} \end{aligned}$$

We have the following pattern of signs : in the positions marked "−", $C_{ij} = -M_{ij}$, and in the positions marked "+", $C_{ij} = M_{ij}$:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

We now write down C, the *matrix of cofactors* of A. The matrix C differs from M by the above pattern of signs. Its entry in the ith row and jth column is the cofactor of $(A)_{ij}$.

$$C = \begin{pmatrix} +(1) & -(2) & +(-6) \\ -(-3) & +(-1) & -(13) \\ +(3) & -(1) & +(-8) \end{pmatrix} = \begin{pmatrix} 1 & -2 & -6 \\ 3 & -1 & -13 \\ 3 & -1 & -8 \end{pmatrix}$$

Step 3 *The Adjoint*

The *adjoint* of A is C^{tr}, the transpose of the matrix of cofactors.

$$\text{adj}(A) = \begin{pmatrix} 1 & 3 & 3 \\ -2 & -1 & -1 \\ -6 & -13 & -8 \end{pmatrix}$$

We now find $A \times \text{adj}(A)$:

$$A \times \text{adj}(A) = \begin{pmatrix} 1 & 3 & 0 \\ 2 & -2 & 1 \\ -4 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 \\ -2 & -1 & -1 \\ -6 & -13 & -8 \end{pmatrix} = \begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{pmatrix} = -5 \times I_3$$

Also $\text{adj}(A) \times A = -5 \times I_3$ (Check).

Thus $\det(A) = -5$ and

$$A^{-1} = -\frac{1}{5} \times \text{adj}(A) = -\frac{1}{5} \begin{pmatrix} 1 & 3 & 3 \\ -2 & -1 & -1 \\ -6 & -13 & -8 \end{pmatrix}$$

<u>The 3×3 Determinant</u>

Let A be a 3×3 matrix. Then $\det(A)$ can be found without computing $\text{adj}(A)$, as follows :

Example 3.3.2 *Let* $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ -2 & 2 & 3 \end{pmatrix}$. *Find* $\det(A)$.

<u>Solution</u>: Let C_{ij} denote the cofactor of the entry $(A)_{ij}$ of A. Recall C_{ij} is given by the determinant of the 2×2 obtained from A by deleting the row and column containing A_{ij}, with possibly a change of sign according to the following pattern:-

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Find the cofactors of the entries in the 1st row of A:

$$C_{11} = +\det \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = 4$$

$$C_{12} = -\det \begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix} = 1$$

$$C_{13} = +\det \begin{pmatrix} -1 & 2 \\ -2 & 2 \end{pmatrix} = 2$$

Then

$$\begin{aligned} &= A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13} \\ &= 2(4) + 1(1) + 3(2) \\ &= 15 \end{aligned}$$

This method of computing the determinant is called *cofactor expansion along the first row.*

$$\det(A) = A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13}$$

<u>Note</u>: We could also do the cofactor expansion along the 2nd row:

$$\det(A) = \underbrace{A_{21}C_{21} + A_{22}C_{22} + A_{23}C_{23}}_{\text{entries of 2nd row of } A \text{ multiplied by their cofactors}}$$

$$\begin{aligned} C_{21} &= -\det\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix} = 3 \\ C_{22} &= +\det\begin{pmatrix} 2 & 3 \\ -2 & 3 \end{pmatrix} = 12 \\ C_{23} &= -\det\begin{pmatrix} 2 & 1 \\ -2 & 2 \end{pmatrix} = -6 \end{aligned}$$

$$\det(A) = -1(3) + 2(12) + 1(-6) = 15$$

In fact, *any* row *or* column of A may be used for the cofactor expansion, so we have six different formulae for $\det(A)$:

$$\begin{aligned} \text{1st row} : \det(A) &= A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13} \\ \text{2nd row} : \det(A) &= A_{21}C_{21} + A_{22}C_{22} + A_{23}C_{23} \\ \text{3rd row} : \det(A) &= A_{31}C_{31} + A_{32}C_{32} + A_{33}C_{33} \end{aligned}$$

$$\begin{aligned} \text{1st column} : \det(A) &= A_{11}C_{11} + A_{21}C_{21} + A_{31}C_{31} \\ \text{2nd column} : \det(A) &= A_{12}C_{12} + A_{22}C_{22} + A_{32}C_{32} \\ \text{3rd column} : \det(A) &= A_{13}C_{13} + A_{23}C_{23} + A_{33}C_{33} \end{aligned}$$

Each of the expressions appearing on the right in the above formulae is a sum of (entry of A) $\times$ (its cofactor) along a particular row or column of A.

Explanation of these formulae: The adjoint of A is the transpose of the matrix of cofactors of A; i.e.

$$\text{adj}(A) = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

$$A \times \text{adj}(A) = \begin{pmatrix} \det(A) & 0 & 0 \\ 0 & \det(A) & 0 \\ 0 & 0 & \det(A) \end{pmatrix} = \text{adj}(A) \times A$$

Working out the entries of $A \times \text{adj}(A)$ on the diagonal gives the first three of our six formulae for $\det(A)$ above. The second three come from $\text{adj}(A) \times A$.

Alternative Method: **"Basket-Weave"** A 3×3 determinant can be found using the "basket-weave" method. For example, let A be the matrix of Example **??**

Step 1 Write down A, and write its 1st 2 columns again on the right :

$$\begin{array}{rrrrr} 2 & 1 & 3 & 2 & 1 \\ -1 & 2 & 1 & -1 & 2 \\ -2 & 2 & 3 & -2 & 2 \end{array}$$

Step 2 $\det(A)$ is given by :

"Sum of products along $\searrow$ diagonals" $-$ "Sum of products along $\swarrow$ diagonals"

$$\begin{aligned} \det(A) &= 2(2)(3) + 1(1)(-2) + 3(-1)(2) - [3(2)(-2) + 2(1)(2) + 1(-1)(3)] \\ &= 12 - 2 - 6 - (-12 + 4 - 3) \\ &= 4 - (-11) \\ &= 15 \end{aligned}$$

Note: The "Basket-Weave" method applies to 3×3 matrices *only*. The cofactor expansion method can be applied to all $n \times n$ matrices for any n.

Summary (of Section **??**) To find the inverse of a 3×3 matrix A :

1. Find $\det(A)$ by cofactor expansion along a row or column, or by the "basket-weave" method.

2. If $\det(A) = 0$, then A has no inverse.

   If $\det(A) \neq 0$, find $\text{adj}(A)$ by calculating all the cofactors of A as in Example **??**

3. The inverse of A is then given by

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$$