

3.2 The Inverse of a Matrix

If we divide a real number by 5 we are multiplying it by $\frac{1}{5}$: $\frac{1}{5}$ is the *reciprocal* or *multiplicative inverse* of 5 in \mathbb{R} . This means

$$\frac{1}{5} \times 5 = 1,$$

i.e., if you multiply 5 by $\frac{1}{5}$, you get 1; multiplying by $\frac{1}{5}$ “reverses” the work of multiplying by 5.

Which $n \times n$ matrices have multiplicative inverses?

Example 3.2.1 Let $A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ and let $B = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$.

Then

$$AB = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$
$$BA = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

So B is an inverse for A .

Definition 3.2.2 Let A be a $n \times n$ matrix. If B is a $n \times n$ matrix for which

$$AB = I_n \quad \text{and} \quad BA = I_n$$

then B is called an inverse for A .

Remarks:

1. Suppose B and C are *both* inverses for a particular matrix A , i.e.

$$BA = AB = I_n \quad \text{and} \quad CA = AC = I_n$$

Then

$$(BA)C = I_n C = C$$

$$\text{Also } (BA)C = B(AC) = BI_n = B$$

Hence $B = C$, and if A has an inverse, its inverse is unique. Thus we can talk about *the* inverse of a matrix.

2. Not every square matrix has an inverse. For example the 2×2 zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ does not.
3. The inverse of a $n \times n$ matrix A , if it exists, is denoted A^{-1} .

Question Given A , how do we

1. Decide if A is *invertible* i.e. if A^{-1} exists?
2. Find A^{-1} ?

The 2×2 Case

Example 3.2.3 Let $A = \begin{pmatrix} 4 & 1 \\ -2 & 3 \end{pmatrix}$.

The *adjoint* of A , denoted $\text{adj}(A)$ is defined as the 2×2 matrix

$$\text{adj}(A) = \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}$$

- obtained from A by

1. Switching the entries 4 and 3 on the main diagonal.
2. Changing the *sign* on the “off-diagonal” entries 1 and -2 .

Now

$$A \times \text{adj}(A) = \begin{pmatrix} 4 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 14 & 0 \\ 0 & 14 \end{pmatrix} = 14 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 14I_2$$

$$\text{adj}(A) \times A = \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 14 & 0 \\ 0 & 14 \end{pmatrix} = 14 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 14I_2$$

So $A \times \text{adj}(A) = \text{adj}(A) \times A = 14I_2$. Then we should define

$$A^{-1} = \frac{1}{14} \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix} = \frac{1}{14} \text{adj}(A).$$

Then $A \times A^{-1} = A^{-1} \times A = I_2$.

Definition: The real number 14 is called the *determinant* of the matrix A .

Definition 3.2.4 If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is any 2×2 matrix, define the adjoint of A by

$$\text{adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

What happens if A is multiplied by its adjoint?

$$\begin{aligned} A \times \text{adj}(A) &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} \\ &= (ad - bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (ad - bc)I_2 \\ \text{adj}(A) \times A &= \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} \\ &= (ad - bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (ad - bc)I_2 \end{aligned}$$

Definition 3.2.5 The real number $ad - bc$ is called the determinant of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and denoted $\det(A)$ (or sometimes $|A|$).

For every 2×2 matrix A we have

$$A \times \text{adj}(A) = \text{adj}(A) \times A = \det(A) \times I_2.$$

Definition 3.2.6 If $\det(A) \neq 0$, we can define the inverse A^{-1} of A by

$$A^{-1} = \frac{1}{\det(A)} \times \text{adj}(A).$$

Then $A \times A^{-1} = A^{-1} \times A = I_2$.

Example 3.2.7 Find the inverses of $A = \begin{pmatrix} 4 & 1 \\ -2 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix}$.

Solution

$$1. A = \begin{pmatrix} 4 & 1 \\ -2 & 2 \end{pmatrix} \implies \text{adj}(A) = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}.$$

$$\det(A) = 4(2) - 1(-2) = 8 + 2 = 10 \quad (\neq 0)$$

$$\implies A^{-1} = \frac{1}{10} \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}.$$

Check: that $A^{-1} \times A = A \times A^{-1} = I_2$.

$$2. B^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} \quad (\text{Exercise}).$$

Notes:

1. If $\det(A) = 0$ for a 2×2 matrix A , then A is *not invertible* - A does not have an inverse. This means there is no 2×2 matrix C for which $AC = CA = I_2$.
2. If $\det(A) \neq 0$, A is called *invertible* or *non-singular*. If $\det(A) = 0$, A is called *non-invertible* or *singular*.

Example 3.2.8 If $A = \begin{pmatrix} 2 & -2 \\ 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}$, find a 2×2 matrix X for which

$$AXB = AB + BA.$$

Solution: We know A and B : we need to isolate X . Observe

$$\det(A) = 2(4) - (-2)(1) = 10 \neq 0$$

$$\det(B) = 3(1) - (-2)(-2) = -1 \neq 0$$

So both A and B are invertible. Multiplying the equation on the left by A^{-1} gives

$$\begin{aligned} A^{-1}(AXB) &= A^{-1}(AB + BA) \\ \implies (A^{-1}A)XB &= (A^{-1}A)B + A^{-1}BA \\ \implies I_2XB &= I_2B + A^{-1}BA \\ \implies XB &= B + A^{-1}BA \end{aligned}$$

Now multiply on the right by B^{-1} :

$$\begin{aligned}(XB)B^{-1} &= (B + A^{-1}BA)B^{-1} \\ \implies X(BB^{-1}) &= BB^{-1} + (A^{-1}BA)B^{-1} \\ \implies XI_2 &= I_2 + A^{-1}BAB^{-1} \\ \implies X &= I_2 + A^{-1}BAB^{-1}\end{aligned}$$

$$\begin{aligned}A^{-1} &= \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{10} \begin{pmatrix} 4 & 2 \\ -1 & 2 \end{pmatrix}, \\ B^{-1} &= \frac{1}{\det(B)} \operatorname{adj}(B) = \frac{1}{(-1)} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -2 & -3 \end{pmatrix}\end{aligned}$$

Then

$$\begin{aligned}A^{-1}BAB^{-1} &= \frac{1}{10} \begin{pmatrix} 4 & 2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ -2 & -3 \end{pmatrix} \\ &= \frac{1}{10} \begin{pmatrix} 8 & -6 \\ -7 & 4 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -9 & -14 \end{pmatrix} \\ &= \frac{1}{10} \begin{pmatrix} 70 & 100 \\ -50 & -70 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ -5 & -7 \end{pmatrix}.\end{aligned}$$

Finally

$$X = I_2 + A^{-1}BAB^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 7 & 10 \\ -5 & -7 \end{pmatrix} = \begin{pmatrix} 8 & 10 \\ -5 & -6 \end{pmatrix}.$$