Chapter 3

Square Matrices : Inverses and Determinants

INTRODUCTION

Definition: A matrix is called *square* if its size is $n \times n$ for some positive integer n (e.g. 2×2, 3×3, etc.).

• Suppose A and B are 3×3 matrices. Then the sum $A + B$ and the products AB and BA are defined, and all of them are 3×3 matrices.

Within the set of $n \times n$ matrices (for a fixed n) we can multiply any matrix by any other, and we stay in the set of $n \times n$ matrices.

• We have defined addition and multiplication for matrices. In this chpater we will consider what it might mean to "divide" by a (square) matrix and when something like this is possible.

• In the real numbers, dividing by a non-zero real number x means multiplying by the reciprocal $1/x$ of x. Before defining something like the reciprocal for matrices, we need a matrix which behaves "something like" the real number 1.

3.1 The $n \times n$ Identity Matrix

Notation: **1.** The set of $n \times n$ matrices with real entries is denoted $M_n(\mathbb{R})$.

2. If A is a $n \times n$ matrix and r is a positive integer, then A^r denotes the $n \times n$ matrix

$$\underbrace{A \times A \times \cdots \times A}_{r}.$$

e.g. in $M_2(\mathbb{R})$, if $A = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$, then

$$A^2 = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 0 & 9 \end{pmatrix}, \text{etc.}$$

Example 3.1.1 $A = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}$ *and let* $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. *Find* AI *and* IA.

Solution:

$$AI = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2(1)+3(0) & 2(0)+3(1) \\ -1(1)+2(0) & -1(0)+2(1) \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} = A$$

$$IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1(2)+0(-1) & 1(3)+0(2) \\ 0(2)+1(-1) & 0(3)+1(2) \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} = A$$

Both AI and IA are equal to A : multiplying A by I (on the left or right) does not affect A.

In general, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is *any* 2×2 matrix, then

$$AI = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$$

and $IA = A$ also.

Definition 3.1.2 $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *is called the* 2×2 identity matrix *(sometimes denoted* I_2*).*

Remarks:

1. The matrix I behaves in $M_2(\mathbb{R})$ like the real number 1 behaves in $\mathbb{R}$ - multiplying a real number x by 1 has no effect on x.

2. The 3×3 identity matrix is $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Check that if A is any 3×3 matrix then $AI_3 = I_3 A = A$.

For any positive integer n, the $n \times n$ *identity matrix* I_n is defined by

$$I_n = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 1 \end{pmatrix}$$

(I_n has 1's along the "main diagonal" and zeroes elsewhere). The entries of I_n are given by :

$$(I_n)_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Theorem 3.1.3 *If A is any $n \times n$ matrix then $AI_n = I_n A = A$*

(i.e. multiplying A on the left or right by I_n leaves A unchanged).
Proof: - that $AI_n = A$.

We show that the entry in the ith row and jth column of the product AI_n is equal to the entry in the ith row and jth column of A.

$$\underbrace{\begin{pmatrix} & & & \\ & & & \\ (A)_{i1} & (A)_{i2} & \dots & (A)_{in} \\ & & & \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} & & & 0 & \\ & & \ddots & \vdots & \\ 0 & \dots & 0 & 1 & 0 \\ & & & \vdots & \\ & & & 0 & \end{pmatrix}}_{I_n} = \underbrace{\begin{pmatrix} & & \vdots & \\ & & \vdots & \\ \dots & \dots & \bullet & \dots \\ & & \vdots & \end{pmatrix}}_{AI_n}$$

$(AI_n)_{ij}$ comes from the ith row of A and the jth column of I_n.

$$\begin{aligned} (AI_n)_{ij} &= (A)_{i1}(0) + (A)_{i2}(0) + \cdots + (A)_{ij}(1) + \cdots + (A)_{in}(0) \\ &= (A)_{ij}(1) \\ &= (A)_{ij} \end{aligned}$$

Thus $(AI_n)_{ij} = (A)_{ij}$ for all i and j - the matrices AI_n and A have the same entries in each position. Then $AI_n = A$.

A similar proof shows $I_n A = A$. □