MATH 1200 - HOMEWORK 18

SOLUTIONS

1. The lines intersect if for some s,t we have

$$[3+t,\ 5+8t,\ -4-3t] = [-6+2s,\ 1-s,\ 7-2s]$$

$$\Rightarrow \quad 3+t = -6+2s \qquad t-2s = -9 \quad (A)$$
$$5+8t = 1-s \qquad 8t+s = -4 \quad (B)$$
$$-4-3t = 7-2s \qquad 3t-2s = -11 \quad (C)$$

From (A) and (B) we have

$$\begin{aligned} t-2s &= -9 \\ 16t+2s &= -8 \end{aligned} \quad \Rightarrow \quad 17t = -17 \quad t = -1$$

$t = -1 \Rightarrow s = 4$

These values also satisfy equation (C), so the lines intersect when $t=-1$, at the point

$$(2, -3, -1)$$

Plane containing L_1 and L_2 has normal vector given by

$$\vec{n} = [1,8,-3] \times [2,-1,-2] = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 8 & -3 \\ 2 & -1 & -2 \end{bmatrix}$$

$$= -19\vec{i} - 4\vec{j} - 17\vec{k}$$
$$= [-19, -4, -17]$$

$\Rightarrow$ Eqn of Plane $\quad 19x + 4y + 17z = d$

$(2,-3,-1)$ belongs to the plane, so

$$d = 19(2) + 4(-3) + 17(-1) = 9$$

$$19x + 4y + 17z = 9$$

2. L Parallel to both P_1 and P_2 $\Rightarrow$ a direction vector for L is given by $\vec{n}_1 \times \vec{n}_2$ where $\vec{n}_1 = [1,-1,2]$ (normal to P_1)

$\vec{n}_2 = [2,-3,1]$ (normal to P_2)

$$\vec{n}_1 \times \vec{n}_2 = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 2 \\ 2 & -3 & 1 \end{bmatrix} = 5\vec{i} + 3\vec{j} - \vec{k}$$

$$= [5,3,-1]$$

$$L \quad \vec{r}(t) = [2,-1,2] + t[5,3,-1], \quad t \in \mathbb{R}$$

3. $$\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 15 \\ 20 \end{pmatrix} = 5\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$\Rightarrow$ corresponding eigenvalue is 5.

4. $$\begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 5 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$\Rightarrow$ corresponding eigenvalue is 5

5. (i) $$A = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$$

$$\lambda I - A = \begin{pmatrix} \lambda - 10 & 9 \\ -4 & \lambda + 2 \end{pmatrix}$$

$$\det(\lambda I - A) = (\lambda - 10)(\lambda + 2) - 9(-4)$$
$$= \lambda^2 - 8\lambda - 20 + 36$$
$$= \lambda^2 - 8\lambda + 16$$

- characteristic polynomial of A.

$(\lambda^2 - 8\lambda + 16) = (\lambda - 4)(\lambda - 4)$

$\Rightarrow \lambda = 4$ is the only eigenvalue of A; it occurs twice.

Eigenvector $\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow 10x - 9y = 4x \qquad 6x - 9y = 0$

$4x - 2y = 4y \qquad 4x - 6y = 0$

Both equations say $2x - 3y = 0$

Thus $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is an eigenvector of A corresponding to $\lambda = 4$

(ii) $B = \begin{pmatrix} -2 & 4 \\ 3 & 2 \end{pmatrix}$

$\lambda I_2 - B = \begin{pmatrix} \lambda+2 & 4 \\ 3 & \lambda-2 \end{pmatrix}$

$\det(\lambda I_2 - B) = (\lambda+2)(\lambda-2) - 12$

$= \lambda^2 - 16$ – characteristic polynomial of B.

$\lambda^2 - 16 = 0 \Rightarrow \lambda = \pm 4$

$\lambda_1 = 4, \lambda_2 = -4$: Eigenvalues of B.

Eigenvectors:

For $\underline{\lambda = 4}$.

$\begin{pmatrix} -2 & 4 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} -2x + 4y = 4x \Rightarrow 4y = 6x \quad 2y = 3x \\ 3x + 2y = 4y \Rightarrow \qquad 2y = 3x \end{matrix}$

$\Rightarrow \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = 4$

For $\underline{\lambda = -4}$

$\begin{pmatrix} -2 & 4 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -4\begin{pmatrix} x \\ y \end{pmatrix} = \begin{matrix} -2x + 4y = -4x \Rightarrow 2x + 4y = 0 \quad x = -2y \\ 3x + 2y = -4y \Rightarrow 3x = -6y \quad x = -2y \end{matrix}$

$\Rightarrow \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -4$.

6. $A = \begin{pmatrix} 0 & -3 & 5 \\ -4 & 4 & 10 \\ 0 & 0 & 4 \end{pmatrix}$

$$\lambda I - A = \begin{pmatrix} \lambda & 3 & -5 \\ 4 & \lambda-4 & -10 \\ 0 & 0 & \lambda-4 \end{pmatrix}$$

$$\begin{aligned} \det(\lambda I - A) &= (\lambda-4)(\lambda(\lambda-4)-3(4)) \\ &= (\lambda-4)(\lambda^2-4\lambda-12) \\ &= \lambda^3-8\lambda^2+4\lambda+48 \end{aligned}$$

(i) Characteristic polynomial of A: $\lambda^3-8\lambda^2+4\lambda+48$

(ii) $$\begin{aligned} \lambda^3-8\lambda^2+4\lambda+48 &= (\lambda+2)(\lambda^2-10\lambda+24) \\ &= (\lambda+2)(\lambda-4)(\lambda-6) \end{aligned}$$

Eigenvalues $-2, 4, 6$

(iii) Eigenvectors.

1. $\lambda = -2$

$$A-(-2I) = \begin{pmatrix} 2 & -3 & 5 \\ -4 & 6 & 10 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\left(\begin{array}{cccc} 2 & -3 & 5 & 0 \\ -4 & 6 & 10 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right) \xrightarrow[\substack{R3\times(1/6) \\ R1\times 1/2}]{R2\times(-1/2)} \left(\begin{array}{cccc} 1 & -3/2 & 5/2 & 0 \\ 2 & -3 & -5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R2\to R2-2R1} \left(\begin{array}{cccc} 1 & -3/2 & 5/2 & 0 \\ 0 & 0 & -10 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow[R2\times(-1/10)]{R1-5/2R3} \left(\begin{array}{cccc} 1 & -3/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$R3 \to R3 - R2$ produces a reduced row echelon form.

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue -2 if

$x = 3/2\, y \qquad z=0 \qquad$ e.g. $\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$

2. $\lambda = 4$

$$A - 4I = \begin{pmatrix} -4 & -3 & 5 \\ -4 & 0 & 10 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc} -4 & -3 & 5 & 0 \\ -4 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R2 \to R2 - R1} \left(\begin{array}{cccc} -4 & -3 & 5 & 0 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R1 \to R1 + R2} \left(\begin{array}{cccc} -4 & 0 & 10 & 0 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[R2 \times (1/3)]{R1 \times (-1/4)} \left(\begin{array}{cccc} 1 & 0 & -5/2 & 0 \\ 0 & 1 & 5/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is an eigenvector of A corresponding to $\lambda = 4$ if $x = 5/2\, z$ and $y = -5/3\, z$

e.g. $\begin{pmatrix} 15 \\ -10 \\ 6 \end{pmatrix}$

3. $\lambda = 6$

$$A - 6I = \begin{pmatrix} -6 & -3 & 5 \\ -4 & -2 & 10 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\left(\begin{array}{cccc} -6 & -3 & 5 & 0 \\ -4 & -2 & 10 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right) \xrightarrow[R3 \times (-1/2)]{R1 \times (-1/6)} \left(\begin{array}{cccc} 1 & 1/2 & -5/6 & 0 \\ -4 & -2 & 10 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R2 \to R2 + 4R1} \left(\begin{array}{cccc} 1 & 1/2 & -5/6 & 0 \\ 0 & 0 & 20/3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow[R2 - 20/3 R3]{R1 + 5/6 R3} \left(\begin{array}{cccc} 1 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$R2 \leftrightarrow R3$ produces a RREF, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is an eigenvector if $x = -1/2\, y$ and $z = 0$, e.g. $\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$

Q6. $B = \begin{pmatrix} 3 & 4 & -1 \\ -1 & -2 & 1 \\ 3 & 9 & 0 \end{pmatrix}$

$$\lambda I - B = \begin{pmatrix} \lambda-3 & -4 & 1 \\ 1 & \lambda+2 & -1 \\ -3 & -9 & \lambda \end{pmatrix}$$

$$\begin{aligned}\det(\lambda I - B) &= (\lambda-3)\left[(\lambda+2)\lambda - 9\right] + 4(\lambda-3) + 1(-9+3(\lambda+2)) \\ &= (\lambda-3)(\lambda^2+2\lambda-9) + 4\lambda - 12 - 9 + 3\lambda + 6 \\ &= \lambda^3 - \lambda^2 - 15\lambda + 27 + 7\lambda - 15 \\ &= \lambda^3 - \lambda^2 - 8\lambda + 12\end{aligned}$$

(i) Characteristic polynomial of B $\lambda^3 - \lambda^2 - 8\lambda + 12$

(ii) $$\begin{aligned}\lambda^3 - \lambda^2 - 8\lambda + 12 &= (\lambda-2)(\lambda^2+\lambda-6) \\ &= (\lambda-2)(\lambda+3)(\lambda-2)\end{aligned}$$

Eigenvalues $\lambda = 2$ (occurring twice), $\lambda = -3$.

(iii) EIGENVECTORS

1. $\lambda = 2$ $B - 2I = \begin{pmatrix} 1 & 4 & -1 \\ -1 & -4 & 1 \\ 3 & 9 & -2 \end{pmatrix}$

$$\left(\begin{array}{cccc} 1 & 4 & -1 & 0 \\ -1 & -4 & 1 & 0 \\ 3 & 9 & -2 & 0 \end{array}\right) \xrightarrow[R3 \to R3 - 3R1]{R2 \to R2 + R1} \left(\begin{array}{cccc} 1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R1 \leftrightarrow R3} \left(\begin{array}{cccc} 1 & 4 & -1 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R2 \times -1/3} \left(\begin{array}{cccc} 1 & 4 & -1 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R1 \to R1 - 4R2} \left(\begin{array}{cccc} 1 & 0 & 1/3 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is an eigenvector if $x = -1/3\, z$
$y = 1/3\, z$

e.g. $\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$

2. $\lambda = -3$

$$B - (-3I) = \begin{pmatrix} 6 & 4 & -1 \\ -1 & 1 & 1 \\ 3 & 9 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 4 & -1 & 0 \\ -1 & 1 & 1 & 0 \\ 3 & 9 & 3 & 0 \end{pmatrix} \xrightarrow[R3 \times 1/3]{R2 \times -1} \begin{pmatrix} 6 & 4 & -1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & 3 & 1 & 0 \end{pmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{pmatrix} 1 & -1 & -1 & 0 \\ 6 & 4 & -1 & 0 \\ 1 & 3 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow[R3 \to R3 - R1]{R2 \to R2 - 6R1} \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & 10 & 5 & 0 \\ 0 & 4 & 2 & 0 \end{pmatrix} \xrightarrow{R2 \times 1/10} \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 4 & 2 & 0 \end{pmatrix} \xrightarrow{R3 \to R3 - 4R2}$$

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R1 \to R1 + R2} \begin{pmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is an eigenvector if $x = \frac{1}{2}z$ and $y = -\frac{1}{2}z$

e.g. $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

Q7. See Lecture Notes Section 7.1.