Section 7.1 : Introduction to Eigenvalues and Eigenvectors

Let A be a 2×2 matrix; for example

$$A = \begin{pmatrix} 2 & 8 \\ 3 & -3 \end{pmatrix}.$$

If $\vec{v}$ is a vector in $\mathbb{R}^2$, e.g. $\vec{v} = [2, 3]$, then we can think of the components of $\vec{v}$ as the entries of a column vector (i.e. a 2×1 matrix). Thus

$$[2, 3] \leftrightarrow \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

If we multiply this vector on the left by the matrix A, we get another column vector with two entries :

$$A \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 & 8 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2(2) + 8(3) \\ 3(2) + (-3)(3) \end{pmatrix} = \begin{pmatrix} 28 \\ -3 \end{pmatrix}$$

So multiplication on the left by the 2×2 matrix A is a function sending the set of 2×1 column vectors to itself - or, if we wish, we can think of it as a function from the set of vectors in $\mathbb{R}^2$ to itself.

Note: In fact this function is an example of a *linear transformation* from $\mathbb{R}^2$ into itself. Linear transformations are functions which have certain interesting geometric properties. Basically they are functions which can be represented in this way by matrices.

In general, if v is a column vector with two entries, then Av is a another vector (with two entries), which typically does not resemble v at all. For example if $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ then

$$Av = \begin{pmatrix} 2 & 8 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 18 \\ -3 \end{pmatrix}$$

However, suppose $v = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$. Then

$$Av = \begin{pmatrix} 2 & 8 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} 8 \\ 3 \end{pmatrix} = \begin{pmatrix} 40 \\ 15 \end{pmatrix} = 5 \begin{pmatrix} 8 \\ 3 \end{pmatrix}$$

i.e. $A \begin{pmatrix} 8 \\ 3 \end{pmatrix} = 5 \begin{pmatrix} 8 \\ 3 \end{pmatrix}$, or

Multiplying the vector $\begin{pmatrix} 8 \\ 3 \end{pmatrix}$ *(on the left) by the matrix* $\begin{pmatrix} 2 & 8 \\ 3 & -3 \end{pmatrix}$ *is the same as multiplying it by* 5.

Terminology: $\begin{pmatrix} 8 \\ 3 \end{pmatrix}$ is called an *eigenvector* for the matrix $A = \begin{pmatrix} 2 & 8 \\ 3 & -3 \end{pmatrix}$ with corresponding *eigenvalue* 5.

Definition 7.1.1: Let A be a $n \times n$ matrix, and let v be a non-zero column vector with n entries (so not all of the entries of v are zero). Then v is called an *eigenvector* for A if

$$A\,v = \lambda\,v,$$

where λ is some real number.

In this situation λ is called an *eigenvalue* for A, and v is said to *correspond* to λ.

Note: "λ" is the symbol for the Greek letter *lambda*. It is conventional to use this symbol to denote an eigenvalue.

Example 7.1.2*: If $A = \begin{pmatrix} -1 & 1 \\ -2 & -4 \end{pmatrix}$ and $v = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, then

$$Av = \begin{pmatrix} -1 & 1 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ -2 \end{pmatrix} = -3v$$

Thus $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is an eigenvector for the matrix $\begin{pmatrix} -1 & 1 \\ -2 & -4 \end{pmatrix}$ corresponding to the eigenvalue -3.

Question: Given a $n \times n$ matrix A, how can we find its eigenvalues and eigenvectors?

Answer: We are looking for column vectors v and real numbers λ satisfying

$$\begin{aligned} Av &= \lambda\,v \\ \text{i.e. } \lambda\,v - Av &= \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \\ \Longrightarrow \lambda I_n v - Av &= \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \\ \Longrightarrow \underbrace{(\lambda I_n - A)}_{\text{a } n\times n \text{ matrix}}\,v &= \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \end{aligned}$$

This may be regarded as a system of linear equations in which the coefficient matrix is $\lambda I_n - A$ and the variables are the n entries of the column vector v, which we can denote by $x_1, \ldots, x_n$. We are looking for solutions to

$$(\lambda I_n - A)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

This system always has at least one solution : namely $x_1 = x_2 = \cdots = x_n = 0$ - all entries of v are zero. However this solution does *not* give an eigenvector since eigenvectors must be non-zero.

The system can have additional solutions only if $\det(\lambda I_n - A) = 0$ (otherwise if the square matrix $\lambda I_n - A$ is invertible, the system will have $x_1 = x_2 = \cdots = x_n = 0$ as its *unique* solution).

Conclusion: The *eigenvalues* of A are those values of λ for which $\det(\lambda I_n - A) = 0$.

Example 7.1.3*: Let $A = \begin{pmatrix} 10 & -8 \\ 4 & -2 \end{pmatrix}$. Find all eigenvalues of A and find an eigenvector corresponding to each eigenvalue.

Solution: We need to find all values of λ for which $\det(\lambda I_2 - A) = 0$.

$$\begin{aligned} \lambda I_2 - A &= \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 10 & -8 \\ 4 & -2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 10 & -8 \\ 4 & -2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda - 10 & 8 \\ -4 & \lambda + 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \det(\lambda I_2 - A) &= (\lambda - 10)(\lambda + 2) - 8(-4) \\ &= \lambda^2 - 10\lambda + 2\lambda - 20 + 32 \\ &= \lambda^2 - 8\lambda + 12 \end{aligned}$$

So $\det(\lambda I_2 - A)$ is a polynomial of degree 2 in λ. The eigenvalues of A are those values of λ for which

$$\det(\lambda I_2 - A) = 0$$

i.e. $\lambda^2 - 8\lambda + 12 = 0 \Longrightarrow (\lambda - 6)(\lambda - 2) = 0, \quad \lambda = 6$ or $\lambda = 2$

Eigenvalues of A : 6,2.

To find an eigenvector of A corresponding to $\lambda = 6$, we need a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ for which

$$
\begin{aligned}
A\begin{pmatrix} x \\ y \end{pmatrix} &= 6\begin{pmatrix} x \\ y \end{pmatrix} \\
\text{i.e.}\begin{pmatrix} 10 & -8 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} &= 6\begin{pmatrix} x \\ y \end{pmatrix} \\
\Longrightarrow \begin{pmatrix} 10x - 8y \\ 4x - 2y \end{pmatrix} &= \begin{pmatrix} 6x \\ 6y \end{pmatrix} \\
\Longrightarrow 10x - 8y = 6x \quad &\text{and} \quad 4x - 2y = 6y
\end{aligned}
$$

Both of these equations say $x - 2y = 0$; hence *any* non-zero vector $\begin{pmatrix} x \\ y \end{pmatrix}$ in which $x = 2y$ is an eigenvector for A corresponding to the eigenvalue 6. For example we can take $y = 1,\ x = 2$ to obtain the eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Exercises:

1. Show that $\begin{pmatrix} 10 & -8 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 6\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
2. Find an eigenvector for A corresponding to the other eigenvalue $\lambda = 2$.