Section 3.3 : Further Ideas from Matrix Algebra

1. General Properties of Matrix Arithmetic

In the following, A, B and C are matrices and at each item in the list we assume that their sizes are such that all indicated additions and multiplications are defined.

1. $A + B = B + A$: Matrix addition is *commutative.*
2. $(A + B) + C = A + (B + C)$: Matrix addition is *associative.*
3. AB is typically not equal to BA, even when both are defined : Matrix multiplication is *not* commutative.
4. $(AB)C = A(BC)$: Matrix multiplication is *associative.*
5. Distributive laws for matrix multiplication over matrix addition:
   $(A + B)C = AC + BC$
   $A(B + C) = AB + AC$

2. Systems of Linear Equations : A Context for matrix Multiplication

Example 3.3.1: Consider the following system of linear equations :

$$\begin{array}{rcrcrcl} 2x & - & y & + & z & = & 1 \\ x & + & 2y & - & z & = & 2 \\ 2x & + & 3y & - & z & = & 4 \end{array}$$

This could be written in terms of column vectors as follows :

$$\begin{pmatrix} 2x & - & y & + & z \\ x & + & 2y & - & z \\ 2x & + & 3y & - & z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

Using the definition of matrix multiplication, we can observe that the left hand side here is the matrix product

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

i.e. the system can be rewritten as the single matrix equation

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

Note: The 3×3 matrix A appearing above is called the *coefficient matrix* of the system. It consists of the first three columns of the augmented matrix.

This formulation of a system of equations in terms of matrices will be useful later.

Example 3.3.2 Consider the system

$$\begin{array}{rcrcl} 2x & + & y & = & 1 \\ x & + & 3y & = & 2 \end{array} \qquad \longleftrightarrow \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Suppose for a second pair of variables s and t that

$$\begin{array}{rcrcr} x & = & 2s & + & 4t \\ y & = & s & - & t \end{array} \qquad i.e. \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}.$$

Then the system can be rewritten in terms of s and t as

$$\begin{array}{rcrcl} 2(2s+4t) & + & (s-t) & = & 1 \\ (2s+4t) & + & 3(s-t) & = & 2 \end{array} \qquad \Longrightarrow \begin{array}{rcrcl} 5s & + & 7t & = & 1 \\ 5s & + & t & = & 2 \end{array}$$

How does the coefficient matrix here depend on the given information? We could rewrite the above line in terms of matrices as

$$\begin{array}{rcl} \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} & = & \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ \Longrightarrow \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} & = & \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ \Longrightarrow \begin{pmatrix} 5 & 7 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} & = & \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ \Longrightarrow \begin{array}{rcr} 5s & + & 7t \\ 5s & + & t \end{array} & \begin{array}{c} = \\ = \end{array} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

$\begin{pmatrix} 5 & 7 \\ 5 & 1 \end{pmatrix}$ is the product $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix}$.

This example shows how matrix multiplication can arise naturally.

3.The Transpose of a Matrix

Definition 3.3.3: Let A be a $m \times n$ matrix. Then the *transpose* of A, denoted by A^{tr} or A^T, is the $n \times m$ matrix defined as follows:

$$
\begin{array}{ccc}
\text{Entries of 1st column of } A^{tr} & : & \text{Entries of 1st row of } A \\
\text{Entries of 2nd column of } A^{tr} & : & \text{Entries of 2nd row of } A \\
\vdots & \vdots & \vdots \\
\text{Entries of } m\text{th column of } A^{tr} & : & \text{Entries of } m\text{th row of } A
\end{array}
$$

Examples:

1. Let $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & 2 \end{pmatrix}$ (2×3). Then $A^{tr} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 3 & 2 \end{pmatrix}$ (3×2).
2. Let $B = \begin{pmatrix} 2 & 1 & 4 \end{pmatrix}$ (1×3). Then $B^{tr} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$ (3×1).
3. Let $C = \begin{pmatrix} 2 & 0 & 4 \\ 0 & -3 & 1 \\ 4 & 1 & 2 \end{pmatrix}$ (3×3). Then $C^{tr} = \begin{pmatrix} 2 & 0 & 4 \\ 0 & -3 & 1 \\ 4 & 1 & 2 \end{pmatrix}$ (3×3).

Note: The matrix C above is equal to its transpose. Matrices with this property are called *symmetric*. Of course a symmetric matrix must be *square* (i.e. $n \times n$: same number of rows as columns), otherwise it will not even have the same size as its transpose.

Remark: If A is a $m \times n$ matrix, then A^{tr} is $n \times m$ and the products AA^{tr} and $A^{tr}A$ are both defined :

$$
\begin{array}{lcccl}
AA^{tr} & : & (m \times \cancel{n}) \times (\cancel{n} \times m) & : & \text{a } m \times m \text{ matrix.} \\
A^{tr}A & : & (n \times \cancel{m}) \times (\cancel{m} \times n) & : & \text{a } n \times n \text{ matrix.}
\end{array}
$$

We see that both AA^{tr} and $A^{tr}A$ are square (though of different sizes unless A itself is square) : in fact both AA^{tr} and $A^{tr}A$ are *symmetric.*

Example 3.3.4*: Let $A = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$. Find AA^{tr} and $A^{tr}A$.

Solution:

1.

$$
\begin{aligned}
AA^{tr} &= \begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ -1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2(2) + (-1)(-1) + (-1)(-1) & 2(1) + (-1)(0) + (-1)1 \\ 1(2) + 0(-1) + 1(1) & 1(1) + 0(0) + 1(1) \end{pmatrix} \\
&= \begin{pmatrix} 6 & 1 \\ 1 & 2 \end{pmatrix}
\end{aligned}
$$

2.

$$\begin{aligned}
A^{tr}A &= \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2(2)+1(1) & 2(-1)+1(0) & 2(-1)+1(1) \\ -1(2)+0(1) & -1(-1)+0(0) & -1(-1)+0(1) \\ -1(2)+1(1) & -1(-1)+1(0) & -1(-1)+1(1) \end{pmatrix} \\
&= \begin{pmatrix} 5 & -2 & -1 \\ -2 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}
\end{aligned}$$

Note that in each case the result is a symmetric matrix.