# Section 3.2 : Matrix Multiplication

The purpose of this section is to describe multiplication of matrices. This is quite a complicated procedure so we begin with a special case.

**Special Case**: Let $A$ be a $m \times n$ matrix and let $B$ be a column vector with $p$ entries (i.e. a $p \times 1$) matrix. Then

1. The product $AB$ is defined only if $p = n$, i.e. only if the number of entries in $B$ is equal to the number of columns in $A$.
2. If $p = n$ then $AB$ is a $m \times 1$ matrix (a column vector with $m$ entries).

$$\text{“}(m \times n) \times (n \times 1)\text{”} = \text{“}m \times 1\text{”}.$$

What are the entries of this product?

**Example 3.2.1*:** Let $A = \begin{pmatrix} 3 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

Then $AB$ is defined and is a $2 \times 1$ matrix. (“$(2\times \not{3}) \times (\not{3} \times 1)$” = “$2 \times 1$”).

What are the entries of $AB$?

1. The first comes from combining the entries of the *first row* of $A$ with those of $B$ in the following way :

   product of 1st entries + product of 2nd entries + product of 3rd entries

$$\begin{aligned} &(3 \times 2) + ((-2) \times 1) + (1 \times 3) \\ &= 6 - 2 + 3 = 7. \end{aligned}$$

2. The second comes from combining the entries of the *second row* of $A$ with those of $B$ in the same way :

$$(2 \times 2) + (1 \times 1) + ((-2) \times 3) = 4 + 1 - 6 = -1.$$

Thus

$$AB = \begin{pmatrix} 3 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \times 2 + (-2) \times 1 + 1 \times 3 \\ 2 \times 2 + 1 \times 1 + (-2) \times 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \end{pmatrix}.$$

**Example 3.2.2***: Let $A = \begin{pmatrix} 2 & -3 \\ 3 & 1 \\ 2 & 4 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Then $AB$ exists and is a $4 \times 1$ matrix.

$$AB = \begin{pmatrix} 2 & -3 \\ 3 & 1 \\ 2 & 4 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2(1) + (-3)(2) \\ 3(1) + 1(2) \\ 2(1) + 4(2) \\ -1(1) + 0(2) \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ 10 \\ -1 \end{pmatrix}.$$

**General Case**: If $A$ is a $m \times p$ matrix and $B$ is a $q \times n$ matrix, then the product $AB$ id defined if and only if $p = q$, i.e. if and only if

$$\text{No. of columns in } A = \text{No. of rows in } B.$$

In this case the size of $AB$ is $m \times n$.

In general the following "cancellation law" holds for the size of matrix products:

$$\text{``}(m \times \not{p}) \times (\not{p} \times n) = m \times n\text{''}.$$

**Definition**: If $A$ is a $m \times p$ matrix and $B$ is a $p \times n$ matrix, then the product $AB$ is defined and is a $m \times n$ matrix in which the entry in the $i$th row and $j$th column is given by combining the entries of the $i$th row of $A$ with those of the $j$th column of $B$ as in Examples 3.2.1* and 3.2.2*.

**Example 3.2.3***: Let $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and let $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$

Find $AB$ and $BA$.

Solution :

1. $A: \; 2 \times 3, \; B: \; 3 \times 2 \Longrightarrow AB$ will be a $2 \times 2$ matrix.

$$\begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2(3) + (-1)(1) + 3(0) & 2(1) + (-1)(-1) + 3(2) \\ 1(3) + 0(1) + (-1)(0) & 1(1) + 0(-1) + (-1)(2) \end{pmatrix}$$
$$= \begin{pmatrix} 5 & 9 \\ 3 & -1 \end{pmatrix}$$

2. $B: \ 3 \times 2, \ A: \ 2 \times 3 \Longrightarrow BA$ will be a $3 \times 3$.

$$BA = \begin{pmatrix} 7 & -3 & 8 \\ 1 & -1 & 4 \\ 2 & 0 & -2 \end{pmatrix}$$

(Exercise)

Note: $BA \neq AB$ : Matrix multiplication is *not* commutative. In this example $AB$ and $BA$ are both defined but do not even have the same size. It is also possible for only one of $AB$ and $BA$ to be defined, for example this will happen if $A$ is $2 \times 4$ and $B$ is $4 \times 3$. Even if $AB$ and $BA$ are both defined and have the same size (for example if both are $3 \times 3$), the two products are typically different.