Second-order self-force: results and prospects

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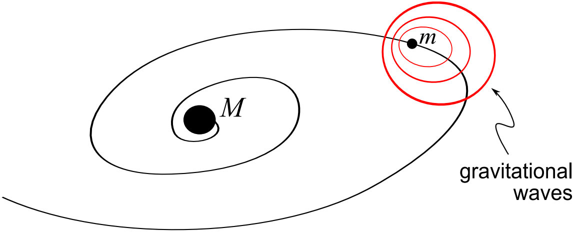
Outline

Outline

Why second order?

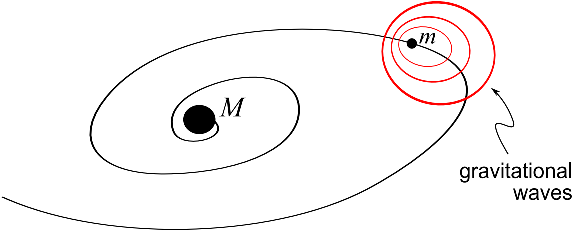


Tracking an inspiral

- inspiral occurs very slowly, on radiation-reaction time $t_{rr} \sim 1/m$
- neglecting second-order self-force leads to error in acceleration $\delta a^\mu \sim m^2$
  $\Rightarrow$ error in position $\delta z^\mu \sim m^2 t^2$
  $\Rightarrow$ after radiation-reaction time $t_{rr} \sim 1/m$, error $\delta z^\mu \sim 1$

$\therefore$ accurately describing orbital evolution requires second-order force

Why second order?



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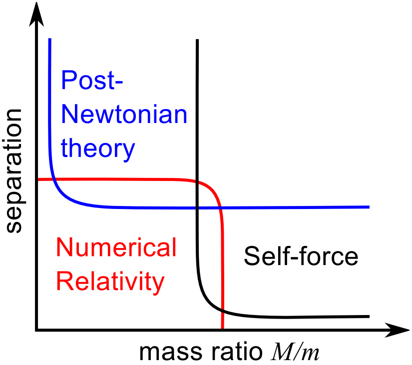
More reasons for second order

Interfacing between models

- establish benchmarks for $m \ll M$ limit of PN and NR
- fix high-order PN parameters
- fix EOB parameters



Modeling IMRIs and similar-mass binaries

- self-force has surprisingly large domain of validity [Le Tiec et al]
- should be highly accurate model for IMRIs
- potentially accurate even for similar-mass binaries

What's required for a second-order approximation scheme?

The physical problem

- small object creates perturbation $\epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + O(\epsilon^3)$ of external background $g_{\alpha\beta}$
- ϵ counts powers of object's mass and size
- must solve Einstein equations

$$\delta G_{\alpha\beta}[h^{(1)}] = 8\pi T^{(1)}_{\alpha\beta}$$
$$\delta G_{\alpha\beta}[h^{(2)}] = 8\pi T^{(2)}_{\alpha\beta} - \delta^2 G_{\alpha\beta}[h^{(1)}]$$

where $\delta^2 G_{\alpha\beta}[h^{(1)}] \sim (\nabla h^{(1)})^2 + h^{(1)} \nabla\nabla h^{(1)}$

Two analytical ingredients needed for solution

1. concrete method of solving EFE for $h^{(n)}_{\alpha\beta}$
2. self-force in terms of $h^{(n)}_{\alpha\beta}$

Apparent obstacles

Solving the EFE: failure of point particle model

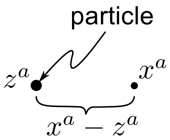
- particle in full spacetime: $T_{\mu\nu} \sim m\frac{\delta^3(x^a - z^a)}{\sqrt{g+h}}$

$$\Rightarrow T^{(2)} \sim mh\delta^3(x^a - z^a) \sim m^2\frac{\delta^3(x^a - z^a)}{x^a - z^a}$$

- also, second-order Einstein tensor

$$\delta^2 G[h^{(1)}] \sim (\partial h^{(1)})^2 \sim 1/(x^a - z^a)^4$$

$\Rightarrow$ seemingly no distributional meaning

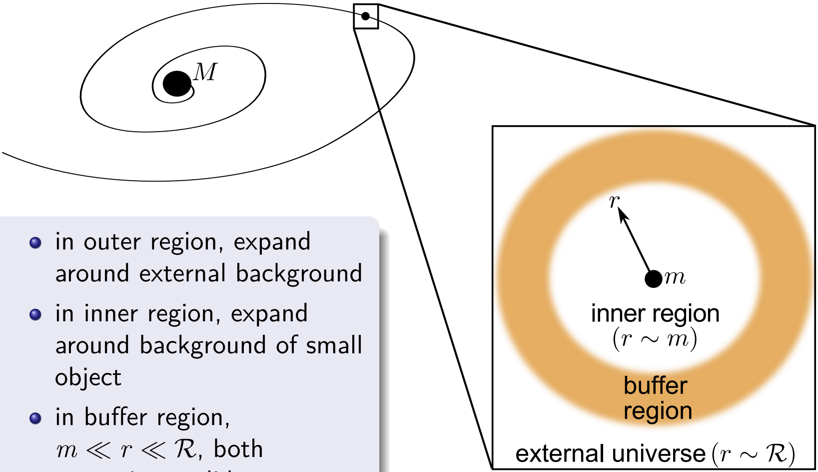


Deriving the self-force: how to define position?

- the mass m must be in some way extended
- how do we pick a "good" representative worldline?

Outline

Matched asymptotic expansions

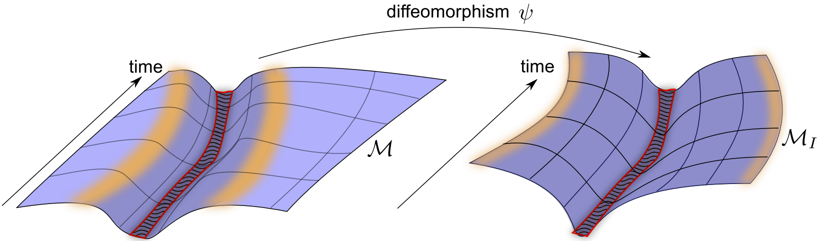


- in outer region, expand around external background
- in inner region, expand around background of small object
- in buffer region, $m \ll r \ll \mathcal{R}$, both expansions valid

Inner expansion

Zoom in on object

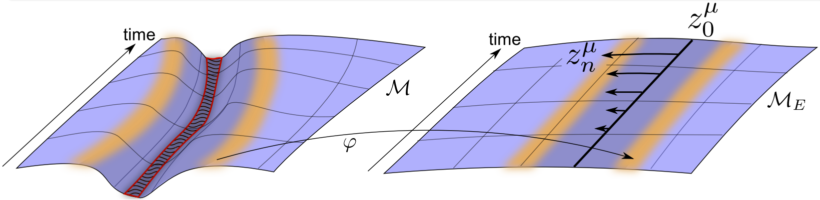
- in inner region, use scaled coords $\tilde{r} \sim r/\epsilon$ to keep size of object fixed, send other distances to infinity as $\epsilon \to 0$
- unperturbed object defines background spacetime $g_{I\mu\nu}$
- buffer region at asymptotic infinity $r \gg m$
  $\Rightarrow$ multipole moments of $g_{I\mu\nu}$ defined there



Outer expansion: Gralla-Wald type ['08]

Expanded worldline

- expand metric in Taylor series
  $\mathsf{g}_{\mu\nu}(x,\epsilon) = g_{\alpha\beta}(x) + \epsilon h^1_{\alpha\beta}(x) + \epsilon^2 h^2_{\alpha\beta}(x) + O(\epsilon^3)$
- expand worldline in Taylor series
  $z^\mu(\tau,\epsilon) = z^\mu_0(\tau) + \epsilon z^\mu_1(\tau) + \epsilon^2 z^\mu_2(\tau) + O(\epsilon^3)$
- z^μ_0: remnant of object at $\epsilon = 0$
- z^μ_n: deviation vectors on z^μ_0



Limitation

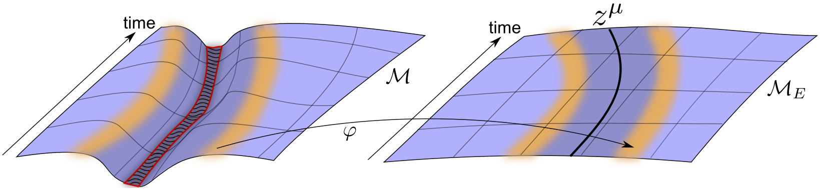
- valid only on timescales $t \sim 1$; much shorter than an inspiral

Outer expansion: self-consistent [Pound '10]

Unexpanded worldline

- rather than finding deviation from z_0^μ, seek a worldline $z^\mu(\tau,\epsilon)$ that faithfully tracks body's bulk motion
- assume generalized expansion of form

$$\mathsf{g}_{\mu\nu}(x,\epsilon) = g_{\mu\nu}(x) + \epsilon h^{(1)}_{\mu\nu}(x;z^\alpha) + \epsilon^2 h^{(2)}_{\mu\nu}(x;z^\alpha) + O(\epsilon^3)$$



Advantage

- potentially accurate on long timescales

Solving the EFE in the buffer region

Expansion of $h^{(n)}_{\mu\nu}$ for small r

- adopt local coordinates centered on a worldline z^μ (or z_0^μ), expand for small r
- inner expansion must not have negative powers of ϵ
  $\Rightarrow$ terms like $\dfrac{\epsilon^n}{r^{n+1}} = \dfrac{1}{\epsilon\tilde{r}^{n+1}}$ not allowed in $\epsilon^n h^{(n)}_{\mu\nu}$

$$\therefore\ h^{(n)}_{\mu\nu} = \frac{1}{r^n} h^{(n,-n)}_{\mu\nu} + r^{-n+1} h^{(n,-n+1)}_{\mu\nu} + r^{-n+2} h^{(n,-n+2)}_{\mu\nu} + \ldots$$

Information from inner expansion

- $1/\tilde{r}^n$ terms arise from large-$\tilde{r}$ expansion of $g_{I\mu\nu}$
  $\Rightarrow$ $h^{(n,-n)}_{\mu\nu}$ is determined by multipole moments of $g_{I\mu\nu}$

Form of solution in buffer region (in Lorenz gauge)

What appears in the solution?

- solve EFE in Lorenz gauge order by order in r
- expand each $h^{(n,p)}_{\mu\nu}$ in spherical harmonics
- given a worldline, the solution at all orders is fully characterized by
  1. body's multipole moments (and corrections thereto): $\sim \frac{Y^{\ell m}}{r^{\ell+1}}$
  2. smooth solutions to vacuum wave equation: $\sim r^{\ell}\, Y^{\ell m}$
- everything else made of (linear or nonlinear) combinations of the above

Self-field and regular field

- multipole moments define $h^{\mathrm{S}(n)}_{\mu\nu}$; interpret as bound field of body
- smooth homogeneous solutions define $h^{\mathrm{R}(n)}_{\mu\nu}$; free radiation, determined by global boundary conditions

First- and second-order solutions in buffer region

First order

- $h^{(1)}_{\mu\nu} = h^{\mathrm{S}(1)}_{\mu\nu} + h^{\mathrm{R}(1)}_{\mu\nu}$
- self-field $h^{\mathrm{S}(1)}_{\mu\nu} \sim 1/r + O(r^0)$ defined by ADM mass m of $g_{I\mu\nu}$
- $h^{\mathrm{R}(1)}_{\mu\nu}$ is undetermined homogenous solution smooth at $r = 0$
- evolution equations: $\dot{m} = 0$ and $a^\mu_{(0)} = 0$ $(a^\mu = a^\mu_{(0)} + \epsilon a^\mu_{(1)} + \ldots)$

Second order

- $h^{(2)}_{\mu\nu} = h^{\mathrm{S}(2)}_{\mu\nu} + h^{\mathrm{R}(2)}_{\mu\nu}$
- $h^{\mathrm{S}(2)}_{\mu\nu} \sim 1/r^2 + O(1/r)$ defined by $m h^{\mathrm{R}(1)}_{\mu\nu}$ and
  1. monopole correction $\delta m_{\mu\nu}$
  2. mass dipole M^μ of $g_{I\mu\nu}$
  3. spin dipole S^μ of $g_{I\mu\nu}$
- evolution equations: $\dot{S}^\mu = 0$, $\delta\dot{m}_{\mu\nu} = \ldots$, and $\ddot{M}^\mu = \ldots$

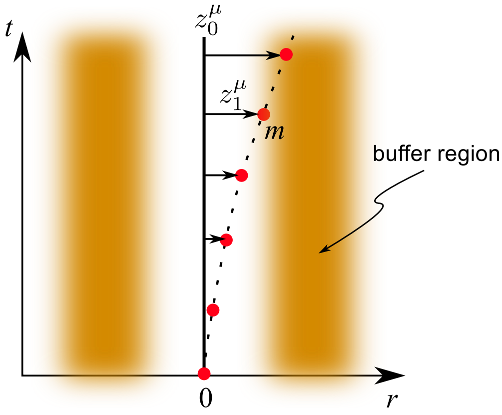
Go to puncture

Outline

Position at first order: Gralla-Wald definition

Reminder: mass dipole

corresponds to displacement of center of mass from origin of coordinates



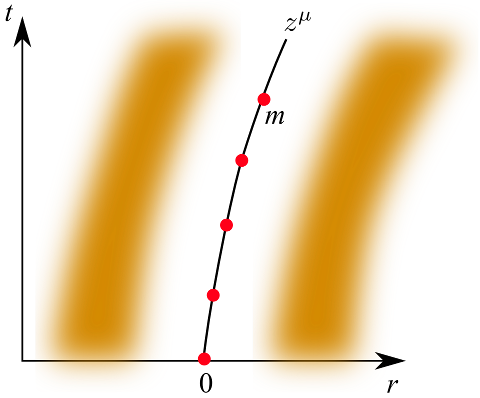
- work in coordinates centered on z_0^μ
- calculate mass dipole M^μ of inner background $g_{I\mu\nu}$
- first-order correction due to self-force:

$$mz_1^\mu = M^\mu$$

Position at first order: self-consistent definition

Mass dipole about z^μ

We want to find worldline z^μ for which $M^\mu = 0$



- work in coordinates centered on unspecified z^μ
- calculate mass dipole M^μ of inner background $g_{I\mu\nu}$
- first-order acceleration of z^μ: whatever ensures $M^\mu \equiv 0$

Proceeding to second order: mass-centered gauges

Problem

- mass dipole moment defined for asymptotically flat spacetimes
- beyond zeroth order, inner expansion is not asymptotically flat

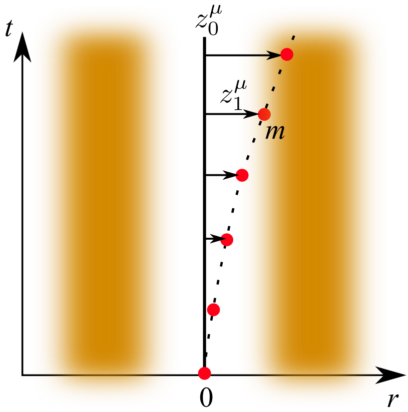
Solution

- find gauge in which field is manifestly mass-centered on z_0^μ (or z^μ)
- define position in other gauges by referring to transformation to that mass-centered gauge

Position at second order: Gralla's definition [2012]

Gauge in a Gralla-Wald-type expansion

On short timescales, position relative to z_0^μ is pure gauge



- start in gauge mass-centered on z_0^μ $\Rightarrow z_1^\mu = z_2^\mu = 0$
- under a small coordinate transformation, the worldline z^μ transforms just as coordinates do
- First order:
$$z_1^\mu = \xi_1^\mu|_{z_0}$$
- Second order:
$$z_2^\mu = \xi_2^\mu|_{z_0} + \xi_1^\nu \partial_\nu \xi_1^\mu|_{z_0}$$

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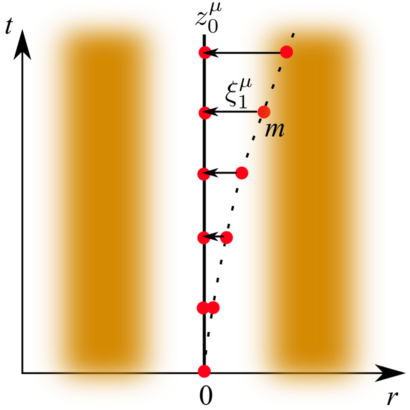

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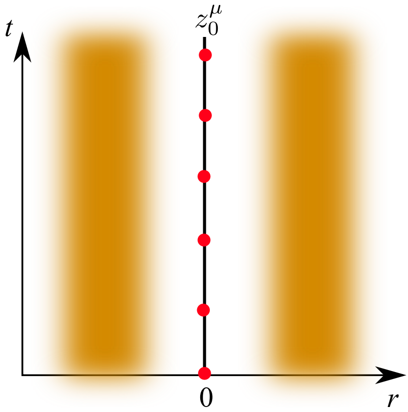

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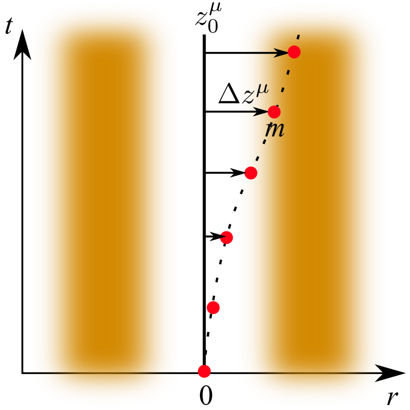

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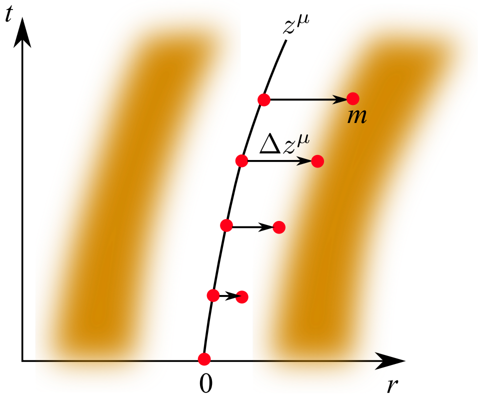
Position at second order: self-consistent [Pound '12]

Gauge in a self-consistet expansion

Over a radiation-reaction time, position relative to z_0^μ is *not* pure gauge



- start in gauge mass-centered on z^μ
- demand that transformation to practical (e.g., Lorenz) gauge does not move z^μ
- i.e., insist
  $$\lim_{r\to 0}\int \xi^a_{(n)}\, d\Omega = 0$$
- ensures worldline in the two gauges is the same

Construction of solution in mass-centered “rest gauge”

Start with a particular inner expansion

- specialize to non-spinning $g_{I\mu\nu}$
- adopt metric of tidally perturbed Schwarzschild black hole
- metric is mass-centered (e.g., $\delta M^\mu = 0$)
  —also in a “rest gauge”: object centered on non-accelerating origin

Generalize the solution

- generalize to any (approximately non-spinning) compact object; i.e., remove boundary conditions specific to BH

Expand in buffer region

- expand at asymptotic infinity (large $\tilde{r}$) and switch to unscaled r

Transforming from rest gauge to Lorenz gauge

Comparison of gauges

- metric in "rest gauge":

$$\mathsf{g}_{tt} \sim \frac{m^2}{r^2} + \frac{m}{r} + r^0(-1) + r^2 e_1(m/r)\tilde{\mathcal{E}}^{\mathsf{q}} + O(r^3)$$

- metric in Lorenz gauge in Fermi coords centered on z^μ:

$$\mathsf{g}_{tt} \sim \frac{m^2}{r^2} + \frac{(m + mh^{\mathrm{R}})}{r} + r^0(-1 + h^{\mathrm{R}} + \text{more}) \\ + r(a_i + \partial h^{\mathrm{R}} + \text{more}) + r^2(\mathcal{E}^{\mathsf{q}} + \partial\partial h^{\mathrm{R}} + \text{more}) + O(r^3)$$

Gauge transformation between them

For a self-consistent solution, seek a unique gauge vector $\epsilon\xi^\mu_{(1)} + \epsilon^2\xi^\mu_{(2)}$ that preserves the position of the worldline

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$$+ \boxed{r(a_i + \partial h^{\rm R} + \text{more})} + r^2(\mathcal{E}^{\mathsf{q}} + \partial\partial h^{\rm R} + \text{more}) + O(r^3)$$

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Self-consistent equation of motion in Lorenz gauge

$$\frac{D^2 z^\mu}{d\tau^2} = \frac{1}{2}\left(g^{\mu\nu} + u^\mu u^\nu\right)\left(g_\nu{}^\rho - h^{\mathrm{R}\,\rho}_\nu\right)\left(h^{\mathrm{R}}_{\sigma\lambda;\rho} - 2h^{\mathrm{R}}_{\rho\sigma;\lambda}\right)u^\sigma u^\lambda + O(\epsilon^3)$$

- here $a^\mu = a^\mu_{(0)} + \epsilon a^\mu_{(1)} + \epsilon^2 a^\mu_{(2)} + \dots$
- and $h^{\mathrm{R}}_{\mu\nu} = \epsilon h^{\mathrm{R}(1)}_{\mu\nu} + \epsilon^2 h^{\mathrm{R}(2)}_{\mu\nu}$

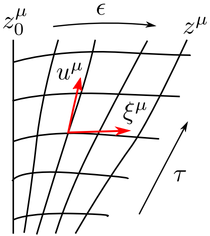
Generalized equivalence principle

- z^μ satisfies geodesic equation in $g_{\mu\nu} + h^{\mathrm{R}}_{\mu\nu}$
- recall: here $g_{\mu\nu} + h^{\mathrm{R}}_{\mu\nu}$ is a "physical" field in the sense of satisfying vacuum EFE
- extends results of Detweiler-Whiting to second order

Gralla-Wald-type equation of motion in Lorenz gauge

Covariant expansion of worldline

- family of worldlines $z^\mu(\tau,\epsilon)$
- tangent vectors: $u^\mu = \frac{dz^\mu}{d\tau}$, $\xi^\mu = \frac{dz^\mu}{d\epsilon}$
- first deviation: $z_1^\mu = \xi^\mu|_{z_0}$
- second deviation: $z_2^\mu = \frac{1}{2}\frac{D\xi^\mu}{d\epsilon}|_{z_0}$



$$\frac{D^2 z_0^\mu}{d\tau^2} = 0$$

$$\frac{D^2 z_1^\mu}{d\tau^2} = R^\mu{}_{\nu\rho\sigma} u_0^\nu u_0^\rho z_1^\sigma - \frac{1}{2} m (g^{\mu\nu} + u_0^\mu u_0^\nu)(2h^{\mathrm{R}(1)}_{\rho\nu;\sigma} - h^{\mathrm{R}(1)}_{\rho\sigma;\nu}) u_0^\rho u_0^\sigma$$

$$\frac{D^2 z_2^\alpha}{d\tau^2} = f_2^\alpha + \frac{1}{2} R^\alpha{}_{\mu\beta\nu;\gamma}(z_1^\mu u_0^\beta z_1^\nu u_0^\gamma - u_0^\mu z_1^\beta u_0^\nu z_1^\gamma) - R^\alpha{}_{\mu\beta\nu}(u_0^\mu z_2^\beta u_0^\nu + 2\dot{z}_1^\mu z_1^\beta u_0^\nu)$$

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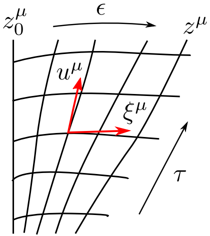

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$$\frac{D^2 z_2^\alpha}{d\tau^2} = f_2^\alpha + \frac{1}{2} R^\alpha{}_{\mu\beta\nu;\gamma}(z_1^\mu u_0^\beta z_1^\nu u_0^\gamma - u_0^\mu z_1^\beta u_0^\nu z_1^\gamma) - R^\alpha{}_{\mu\beta\nu}(u_0^\mu z_2^\beta u_0^\nu + 2\dot{z}_1^\mu z_1^\beta u_0^\nu)$$

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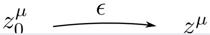

$$f_2^\mu = \frac{1}{2}P_0^{\mu\nu}\left(h^{\mathrm{R}(2)}_{\sigma\lambda;\rho} - 2h^{\mathrm{R}(2)}_{\rho\sigma;\lambda}\right)u_0^\sigma u_0^\lambda - P_0^{\mu\nu}h_\nu^{\mathrm{R}(1)\rho}\left(h^{\mathrm{R}(1)}_{\sigma\lambda;\rho} - 2h^{\mathrm{R}(1)}_{\rho\sigma;\lambda}\right)u_0^\sigma u_0^\lambda$$
$$+\left(h^{\mathrm{R}(1)}_{\sigma\lambda;\nu} - 2h^{\mathrm{R}(1)}_{\nu\sigma;\lambda}\right)\left[\left(\dot z_1^\mu u_0^\nu + u_0^\mu \dot z_1^\nu\right)u_0^\sigma u^\lambda + P_0^{\mu\nu}\left(\dot z_1^\sigma u_0^\lambda + u_0^\sigma \dot z_1^\lambda\right)\right]$$

- second deviation: $z_2^\mu = \frac{1}{2}\frac{D^2 z}{d\epsilon}|_{z_0}$

$$\frac{D^2 z_0^\mu}{d\tau^2} = 0$$
$$\frac{D^2 z_1^\mu}{d\tau^2} = R^\mu{}_{\nu\rho\sigma}u_0^\nu u_0^\rho z_1^\sigma - \tfrac{1}{2}m(g^{\mu\nu} + u_0^\mu u_0^\nu)(2h^{\mathrm{R}(1)}_{\rho\nu;\sigma} - h^{\mathrm{R}(1)}_{\rho\sigma;\nu})u_0^\rho u_0^\sigma$$

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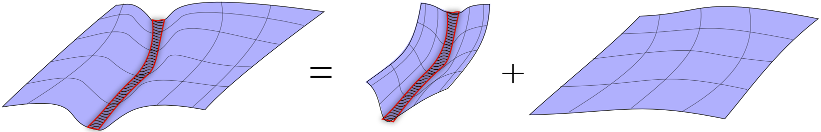
Outline

Effective interior metric

From self-field to singular field

- $h^{\mathrm{S}}_{\mu\nu}$ and $h^{\mathrm{R}}_{\mu\nu}$ derived only in buffer region
- simply extend them to all $r > 0$ (and $r = 0$, for $h^{\mathrm{R}}_{\mu\nu}$)
- does not change field in buffer region or beyond



full metric $g_{\mu\nu}$ = "self field" $h^{\mathrm{S}}_{\mu\nu}$ + effective metric $g_{\mu\nu} + h^{\mathrm{R}}_{\mu\nu}$

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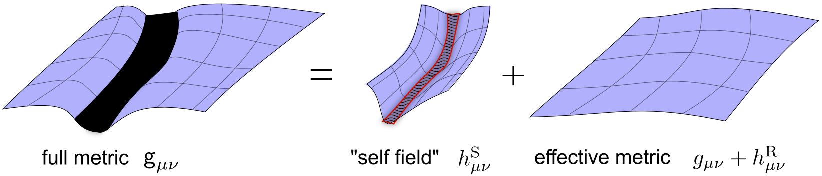

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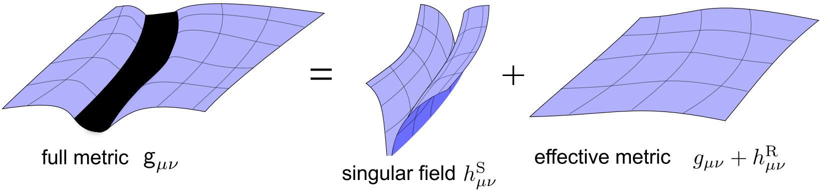
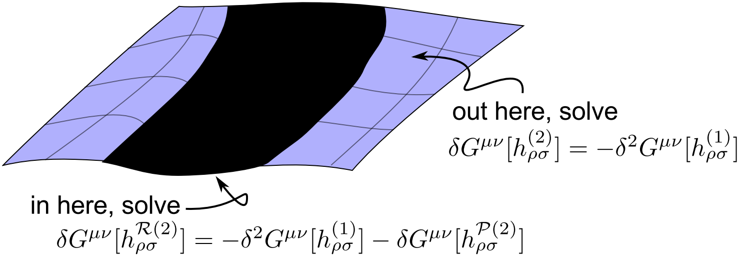
- simply extend them to all $r>0$ (and $r=0$, for $h^{\rm R}_{\mu\nu}$)
- does not change field in buffer region or beyond



Obtaining global solution

Puncture/effective source scheme

- define $h^{\mathcal{P}}_{\mu\nu}$ as small-r expansion of $h^{\rm S}_{\mu\nu}$ truncated at finite order in r
- define $h^{\mathcal{R}}_{\mu\nu} = h_{\mu\nu} - h^{\mathcal{P}}_{\mu\nu} \simeq h^{\rm R}_{\mu\nu}$



The point...

- to calculate effective metric "inside" body and full metric everywhere else, all you need is $h^{\rm S}_{\mu\nu}$ found in buffer region

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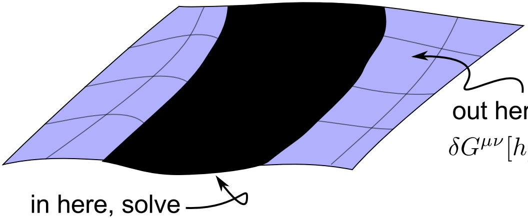


out here, solve

$$\delta G^{\mu\nu}[h^{(2)}_{\rho\sigma}] = -\delta^2 G^{\mu\nu}[h^{(1)}_{\rho\sigma}]$$

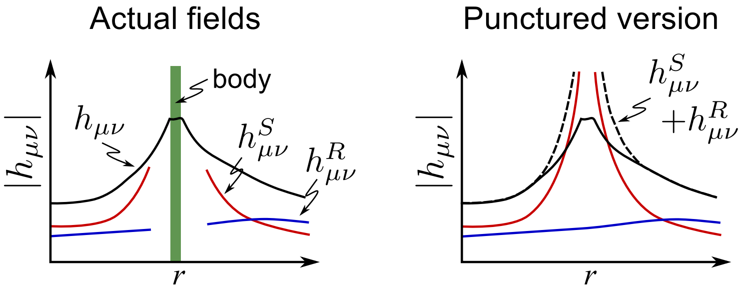
in here, solve

$$\delta G^{\mu\nu}[h^{\mathcal{R}(2)}_{\rho\sigma}] = -\delta^2 G^{\mu\nu}[h^{(1)}_{\rho\sigma}] - \delta G^{\mu\nu}[h^{\mathcal{P}(2)}_{\rho\sigma}]$$

The point...

- to calculate effective metric "inside" body and full metric everywhere else, all you need is $h^{\mathrm{S}}_{\mu\nu}$ found in buffer region

More on puncturing



A note on singularities

- derivations of self-force from matched expansions yield an expression for the force in terms of a manifestly finite field outside the object
- we don't begin with an infinity and subtract an infinity —we write a known finite field as the difference between two known divergent fields

Self-consistent puncture scheme

Let Γ be worldtube around object

and $$h^{\mathcal{R}(n)}_{\mu\nu} = \begin{cases} h^{(n)}_{\mu\nu} & \text{outside } \Gamma \\ h^{(n)}_{\mu\nu} - h^{\mathcal{P}(n)}_{\mu\nu} & \text{inside } \Gamma \end{cases}$$

Simultaneously solve coupled system

$$\Box h^{\mathcal{R}(1)}_{\mu\nu} = \begin{cases} 0 & \text{outside } \Gamma \\ -\Box h^{\mathcal{P}(1)}_{\mu\nu} & \text{inside } \Gamma \end{cases}$$

$$\Box h^{\mathcal{R}(2)}_{\mu\nu} = \begin{cases} -2\delta^2 R_{\mu\nu}[h^{(1)}] & \text{outside } \Gamma \\ -2\delta^2 R_{\mu\nu}[h^{(1)}] - \Box h^{\mathcal{P}(2)}_{\mu\nu} & \text{inside } \Gamma \end{cases}$$

$$\frac{D^2 z^\mu}{d\tau^2} = \frac{1}{2}\left(g^{\mu\nu} + u^\mu u^\nu\right)\left(g_\nu{}^\rho - h^{\mathcal{R}}_\nu{}^\rho\right)\left(h^{\mathcal{R}}_{\sigma\lambda;\rho} - 2h^{\mathcal{R}}_{\rho\sigma;\lambda}\right) u^\sigma u^\lambda,$$

- $h^{\mathcal{P}(2)}_{\mu\nu}$ known analytically in Lorenz gauge [Pound '10, '12]
- puncture moves on z^μ

Gralla-Wald-type puncture scheme

Solve sequence of equations

1. $$\frac{D^2 z_0^\mu}{d\tau^2} = 0$$
2. $$\Box h^{\mathcal{R}(1)}_{\mu\nu} = \begin{cases} 0 & \text{outside } \Gamma_0 \\ -\Box h^{\mathcal{P}(1)}_{\mu\nu} & \text{inside } \Gamma_0 \end{cases}$$
3. $$\frac{D^2 z_1^\mu}{d\tau^2} = R^\mu{}_{\nu\rho\sigma} u_0^\nu u_0^\rho z_1^\sigma - \frac{1}{2} m (g^{\mu\nu} + u_0^\mu u_0^\nu)(2h^{\mathcal{R}(1)}_{\rho\nu;\sigma} - h^{\mathcal{R}(1)}_{\rho\sigma;\nu}) u_0^\rho u_0^\sigma$$
4. $$\Box h^{\mathcal{R}(2)}_{\mu\nu} = \begin{cases} -2\delta^2 R_{\mu\nu}[h^{(1)}] & \text{outside } \Gamma_0 \\ -2\delta^2 R_{\mu\nu}[h^{(1)}] - \Box h^{\mathcal{P}(2)}_{\mu\nu} & \text{inside } \Gamma_0 \end{cases}$$
5. $$\frac{D^2 z_2^\alpha}{d\tau^2} = f_2^\alpha + R \text{ and } \nabla R \text{ terms}$$

- $h^{\mathcal{P}(2)}_{\mu\nu}$ known analytically in Lorenz gauge [Pound '10,'12] and 'P-smooth' gauges [Gralla '12]
- puncture moves on z_0^μ

Outline

The two necessary ingredients

1. Method of solving EFE numerically

- puncture/effective-source scheme [Detweiler '12, Pound '12, Gralla '12]
- puncture known explicitly in Lorenz gauge [Pound '10, '12] and 'P-smooth' gauges [Gralla '12]

2. Equation of motion & definition of worldline

- self-consistent formulation in Lorenz gauge [Pound '12]
- Gralla-Wald-type formulation in 'P-smooth' gauges [Gralla '12] and Lorenz gauge [Pound '13]
- in 'Fermi' gauge (though w/o clear definition of worldline) [Rosenthal '06]

Transforming to a more practical puncture

Punctures in Lorenz and ‘P-smooth’ gauges are written in local coordinates (t, x^a) centered on z^μ or z_0^μ

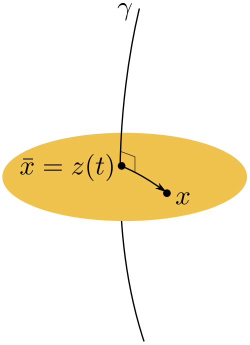
- impractical for numerical calculations in global coordinates

From local coords to covariant expansion

- use puncture in Fermi coordinates
- write tensor in index-free notation

$$h^{\mathcal{P}}(x) = h^{\mathcal{P}}_{tt}(t, x^i)\,dt dt + 2h^{\mathcal{P}}_{ta}(t, x^i)\,dt dx^a + h^{\mathcal{P}}_{ab}(t, x^i)\,dx^a dx^b$$

- express in covariant quantities:
  - $t \to \bar{x}$
  - $x^i \to -e^i_{\bar{\alpha}} \nabla^{\bar{\alpha}} \sigma(x, \bar{x})$
  - $dt,\ dx^a \to$ combinations of $\sigma,\ u^{\bar{\alpha}},\ e^a_{\bar{\alpha}}$



A practical puncture

From covariant expansion to coordinate expansion

- Expand covariant quantities in coordinate differences $\delta x^{\alpha} = x^{\alpha} - x^{\alpha'}$
  - $\sigma^{\alpha'} = -\delta x^{\alpha} + O(\delta x)^2$
  - $g^{\alpha'}_{\beta} = \delta^{\alpha'}_{\beta} + O(\delta x)$
- obtain puncture in, e.g., Schwarzschild or Boyer-Lindquist coordinates Go to puncture
- in principle, second-order puncture scheme (self-consistent or Gralla-Wald type) can be immediately implemented in time domain

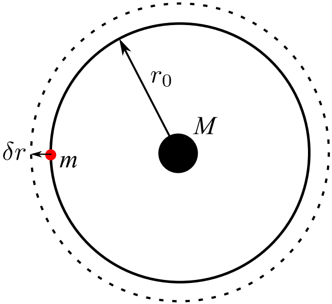
Obstacle to implementation

Even at first order, puncture scheme in time domain suffers from unresolved problem of growing gauge modes

Second-order puncture scheme in frequency domain

Problem tractable in frequency domain

- second-order conservative effects on circular orbits



- use Gralla-Wald-type puncture scheme
- conservative shift in position is simply shift in radius
- can calculate short-term effects
  - $h^R_{\mu\nu} u^\mu u^\nu$
  - z_2^μ, second-order shift in position
  - EOB parameters
- calculation underway w/ Barack, Warburton, Wardell

Conclusion

Benefits of second order

- necessary to model inspiral
- complements and advances PN/NR/EOB

Results

- second-order puncture
- second-order equation of motion

Prospects

- time domain: major obstacle at first order
- frequency domain: calculations of short-term effects should soon be achieved

Longer-term goals

- self-consistent evolution or good alternative to it for inspiral

$h^{(1)}h^{(1)}$ terms in $h^{S(2)}$, Fermi coordinates

$$
\begin{aligned}
\bar{h}^{Stt}_{(2)} = {}& \frac{3m^2}{r^2} - \frac{m}{r}\bar{h}^{Rij}_{(1)}\hat{n}_{ij} - m\left(\tfrac{11}{5}\bar{h}^{Rb}_{(1)a,b} + \tfrac{1}{10}\bar{h}^{Rb}_{(1)b,a} + \bar{h}^{Rt}_{(1)a,t} - \tfrac{3}{2}\bar{h}^{Rtt}_{(1),a}\right)n^a \\
& - \tfrac{7}{3}m^2\mathcal{E}^{ab}\hat{n}_{ab} - \tfrac{1}{2}m\bar{h}^{Rab,c}_{(1)}\hat{n}_{abc} \\
& + r\Bigg[\tfrac{1}{270}m\Big(-252\bar{h}^{Rab}_{(1),ab} + 84\bar{h}^{Rb}_{(1)b}{}^{,a}{}_{a} - 268\mathcal{E}^{ab}\bar{h}^{R}_{(1)ab} + 630\bar{h}^{Rtb}_{(1),bt} \\
& - 15\bar{h}^{Rb}_{(1)b,tt} + 675\bar{h}^{Rtt}_{(1),tt}\Big) + \tfrac{23}{9}m\mathcal{E}^{ab}\bar{h}^{Rc}_{(1)b}\hat{n}_{ac} + \tfrac{5}{9}m\mathcal{B}^{ac}\epsilon_{bcd}\bar{h}^{Rtb}_{(1)}\hat{n}_a{}^d \\
& + \tfrac{1}{72}m\Big(108\bar{h}^{Rtt,ab}_{(1)} + \mathcal{E}^{ab}\big(96\bar{h}^{Rtt}_{(1)} - 76\bar{h}^{Rc}_{(1)c}\big)\Big)\hat{n}_{ab} \\
& + \tfrac{1}{42}m\Big(26\bar{h}^{R}_{(1)ab}{}^{,c}{}_{c} - 78\bar{h}^{Rc}_{(1)b,ac} - 9\bar{h}^{Rc}_{(1)c,ba} - 21\bar{h}^{Rt}_{(1)b,at} - 7\bar{h}^{R}_{(1)ab,tt}\Big)\hat{n}^{ab} \\
& - \tfrac{29}{20}m^2\mathcal{E}^{abc}\hat{n}_{abc} + \tfrac{1}{6}m\Big(-2\bar{h}^{Rab}_{(1),cd} + 7\mathcal{E}^{ba}\bar{h}^{Rcd}_{(1)}\Big)\hat{n}_{abcd}\Bigg] + O(r^2)
\end{aligned}
$$

Return

$h^{S(1)}h^{S(1)}$ terms, covariant puncture

$$
\begin{aligned}
h^{\mathcal{P}(2)}_{\alpha\beta} = {} & \frac{m^2 g^{\alpha'}_{\mu} g^{\beta'}_{\nu}}{\mathsf{s}^4}\left(5\mathsf{s}^2 g_{\alpha'\beta'} - 14\mathsf{r}\sigma_{(\alpha'}u_{\beta')} - 7\mathsf{r}^2 u_{\alpha'}u_{\beta'} + 3\mathsf{s}^2 u_{\alpha'}u_{\beta'} - 7\sigma_{\alpha'}\sigma_{\beta'}\right) \\
& + \frac{m^2 g^{\alpha'}_{\mu} g^{\beta'}_{\nu}}{150\mathsf{s}^6}\Big[10\mathsf{s}^4 R_{\alpha'\sigma\beta'\sigma} + 20\mathsf{r}\mathsf{s}^4 R_{(\alpha'|u|\beta')\sigma} + \mathsf{s}^4(10\mathsf{r}^2 + 52\mathsf{s}^2)R_{\alpha'u\beta'u} \\
& - 350\mathsf{r}\mathsf{s}^2\sigma_{(\alpha'}R_{\beta')\sigma u\sigma} - 350\mathsf{r}^2\mathsf{s}^2 u_{(\alpha'}R_{\beta')\sigma u\sigma} + 170\mathsf{s}^4 u_{(\alpha'}R_{\beta')\sigma u\sigma} \\
& + 700\mathsf{r}^2\mathsf{s}^2\sigma_{(\alpha'}R_{\beta')u\sigma u} - 620\mathsf{r}\mathsf{s}^4 u_{(\alpha'}R_{\beta')u\sigma u} + 700\mathsf{r}^3\mathsf{s}^2 u_{(\alpha'}R_{\beta')u\sigma u} \\
& + 1120R_{u\sigma u\sigma}\mathsf{r}\mathsf{s}^2\sigma_{(\alpha'}u_{\beta')} + 1060R_{u\sigma u\sigma}\mathsf{r}^2\mathsf{s}^2 u_{\alpha'}u_{\beta'} - 700R_{u\sigma u\sigma}\mathsf{r}^2\sigma_{\alpha'}\sigma_{\beta'} \\
& - 1400R_{u\sigma u\sigma}\mathsf{r}^3\sigma_{(\alpha'}u_{\beta')} - 700R_{u\sigma u\sigma}\mathsf{r}^4 u_{\alpha'}u_{\beta'} + 210R_{u\sigma u\sigma}\mathsf{s}^2\sigma_{\alpha'}\sigma_{\beta'} \\
& + 120R_{u\sigma u\sigma}\mathsf{s}^4 u_{\alpha'}u_{\beta'} + g_{\alpha'\beta'}\left(250\mathsf{r}^2\mathsf{s}^2 + 10\mathsf{s}^4\right)R_{u\sigma u\sigma}\Big] \\
& - \frac{16}{15}m^2\ln(\mathsf{s})g^{\alpha'}_{\mu} g^{\beta'}_{\nu} R_{\alpha'u\beta'u} \\
& + \text{order } \sqrt{\sigma} \text{ terms}
\end{aligned}
$$

$h^{\mathrm{S}(1)}h^{\mathrm{S}(1)}$ terms, circular orbits in Schwarzschild coordinates

$$
\begin{aligned}
h_{tt}^{\mathcal{P}(2)} = {} & \frac{m^2\left[(3E^2-5)r_0+10M\right]}{\rho^2 r_0} - \frac{28\delta Q^2 E^4 m^2 r_0^6 \Omega^2}{\rho^4 r_0^2 f_0^2} \\
& - \frac{\delta r m^2}{\rho^4 r_0^4 f_0^3}\left\{8\delta Q^2 E^2 r_0^5 \Omega^2\left[(20-13E^2)Mr_0 + 5(2E^2-1)r_0^2 - 20M^2\right]\right. \\
& + r_0 f_0\left[(3E^2-5)r_0+10M\right]\left(16\delta Q^2 M^2 r_0 - \delta r^2 M + \delta\theta^2 r_0^3 f_0^2\right. \\
& \left.\left. -16\delta Q^2 M r_0^2 + 4\delta Q^2 r_0^3\right)\right\} + \frac{2\delta r m^2 M\left[(3E^2-5)r_0+10M\right]}{\rho^2 r_0^3 f_0} \\
& + \frac{56\delta Q^2 \delta r E^4 m^2 r_0^6 \Omega^2}{\rho^6 r_0^4 f_0^4}\left\{r_0^3\left[\delta\theta^2 + \delta Q^2(8E^2 r_0^2\Omega^2+4)\right]\right. \\
& \left. +4M^2 r_0(\delta\theta^2+4\delta Q^2) - M\left[\delta r^2 + 4r_0^2(\delta\theta^2+4\delta Q^2)\right]\right\} \\
& + \text{order } (\delta x^\alpha)^0 \text{ terms} + \text{order } \delta x^\alpha \text{ terms}
\end{aligned}
$$

Return