A q-MULTISUM IDENTITY ARISING FROM FINITE CHAIN RING PROBABILITIES

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ABSTRACT. In this note, we prove a general identity between a q-multisum $B_N(q)$ and a sum of N^2 products of quotients of theta functions. The q-multisum $B_N(q)$ recently arose in the computation of a probability involving modules over finite chain rings.

1. Introduction

Probabilistic proofs of classical q-series identities constitute an intriguing part of the literature in combinatorics. A prominent example of this perspective concerns the Andrews-Gordon identities [1, 8] which state for $1 \leq i \leq k$ and $k \geq 2$

$$\sum_{n_1,\dots,n_{k-1}\geq 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}}}{(q)_{n_1}\cdots(q)_{n_k}} = \prod_{\substack{s=1 \\ s\not\equiv 0,\pm i \pmod{2k+1}}}^{\infty} \frac{1}{1-q^s}, \tag{1.1}$$

where $N_j = n_j + \cdots + n_{k-1}$. Here and throughout, we use the standard q-hypergeometric (or "q-Pochhammer symbol") notation

$$(a)_n = (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k),$$

valid for $n \in \mathbb{N} \cup \{\infty\}$. In [7], Fulman uses a Markov chain on the nonnegative integers to prove the extreme cases $i = 1$ and $i = k$ of (1.1). Chapman [3] cleverly extends Fulman's methods to prove (1.1) in full generality. In [4], Cohen explicitly computes probability laws of p^ℓ-ranks of finite abelian groups to give a group-theoretic proof of (1.1). For a generalization of this computation, see [5]. In this note, we are interested in a recent probability computation with a ring-theoretic flavor as it leads to an expression similar to the left-hand side of (1.1).

Let $\mathcal{R}$ be a finite chain ring with radical $\mathcal{N}$, q be the order of the residue field $\mathcal{R}/\mathcal{N}$ and N be the index of nilpotency of $\mathcal{N}$ (for further details, see Section 2 in [10]). Recently, the authors of [2] expressed the density $\psi(n,k,q,N)$ of free submodules $\mathcal{M}$ of $\mathcal{R}^n$ (over $\mathcal{R}$) of type $k := \log_{|\mathcal{R}|}(|\mathcal{M}|)$ as $n \to \infty$ as the reciprocal of the q-multisum (replacing $1/q$ in their notation with q)

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$$B_N(q) := \sum_{\substack{K_2,\ldots,K_N \geq 0 \\ N \mid K_2+\cdots+K_N}} \frac{q^{K_2^2+\cdots+K_N^2-(K_2+\cdots+K_N)^2/N}}{(q)_{k_2}\cdots(q)_{k_N}}, \tag{1.2}$$

where $N \geq 2$ is an integer and $K_i = \sum_{j=2}^{i} k_j$. Upper and lower bounds for $B_N(q)$ are obtained and then used to show (under suitable conditions) that $\psi(n,k,q,N)$ is at least $1-\epsilon$ where $0<\epsilon<1$ (see Theorems 7 and 9, respectively, in [2]). Moreover, we have

$$B_2(q) = \prod_{\substack{s=1 \\ s \equiv \pm 2, \pm 3, \pm 4, \pm 5 \pmod{16}}}^{\infty} \frac{1}{1-q^s}, \tag{1.3}$$

which is (S.83) in [13]. In view of (1.1) and (1.3), the authors in [2] posed the following (slightly rewritten) problem.

Problem 1.1. *Determine whether $B_N(q)$ can be expressed as a product of q-Pochhammer symbols.*

The purpose of this note is to solve Problem 1.1. It turns out that the solution is slightly more involved than either (1.1) or (1.3), namely $B_N(q)$ is a sum of N^2 products of quotients of theta functions (but not a single product of q-Pochhammer symbols, for general N). Before stating our main result, we recall some further standard notation:

$$\begin{aligned} j(x;q) &:= (x)_\infty (q/x)_\infty (q)_\infty, \\ j(x_1,x_2,\ldots,x_n;q) &:= j(x_1;q)j(x_2;q)\cdots j(x_n;q), \\ J_{a,m} &:= j(q^a;q^m), \\ \overline{J}_{a,m} &:= j(-q^a;q^m), \\ J_m &:= J_{m,3m} = (q^m;q^m)_\infty. \end{aligned}$$

Note that these quantities are products of q-Pochhammer symbols. Our main result is now the following.

Theorem 1.2. *For all $N \geq 2$, we have*

$$B_N(q) = \frac{1}{(q)_\infty^2 \overline{J}_{0,N(N+2)}} \sum_{r=0}^{N-1}\sum_{s=0}^{N-1} \frac{(-1)^{r+s+1} q^{\binom{r}{2}+\binom{s+1}{2}+r(s+1)(N+1)+r+s+1} J^3_{N^2(N+2)}}{j((-1)^N q^{N(N+2)r+N(N+3)/2}; q^{N^2(N+2)})} \times \frac{j(-q^{N(s-r)};q^{N^2}) j(q^{N(N+2)(r+s)+N(N+3)};q^{N^2(N+2)})}{j((-1)^N q^{N(N+2)s+N(N+3)/2}; q^{N^2(N+2)})}. \tag{1.4}$$

Formula (1.4) is very efficient in practice for computing explicit values of $B_N(q)$. Indeed, contrarily to (1.2) which requires computing a $(N-1)$-fold sum, (1.4) only involves a double sum. As a comparison with Table 1 in [2], we compute $B_N(q)$ for $2 \leq N \leq 10$ and $N=100$ and $1/q = 2,3,5,7,11$ to five decimals with Maple using (1.4). Table 1 below suggests that when $q \to 0$, the limiting value of $B_N(q)$ is 1. This statement is confirmed in [2, Corollary 11, (1)].

The paper is organized as follows. In Section 2, we recall one of the main results from [14], then prove Theorem 1.2. In Section 3, we make some concluding remarks.

$N \setminus 1/q$	2	3	5	7	11
2	0.59546	0.84191	0.95049	0.97627	0.99092
3	0.47084	0.79666	0.94102	0.97295	0.99010
4	0.42109	0.78230	0.93915	0.97248	0.99002
5	0.39877	0.77759	0.93877	0.97241	0.99002
6	0.38819	0.77603	0.93870	0.97240	0.99002
7	0.38304	0.77551	0.93868	0.97240	0.99002
8	0.38050	0.77533	0.93868	0.97240	0.99002
9	0.37924	0.77528	0.93868	0.97240	0.99002
10	0.37861	0.77526	0.93868	0.97240	0.99002
100	0.37798	0.77525	0.93868	0.97240	0.99002
$(q)_\infty$	0.28879	0.56013	0.76033	0.83680	0.90083

TABLE 1. Values of $B_N(q)$

2. PROOF OF THEOREM 1.2

Before the proof of Theorem 1.2, we need to recall some background from the important work of Hickerson and Mortenson [14]. First, we employ the Hecke-type series

$$f_{a,b,c}(x,y,q) := \Big(\sum_{r,s\geq 0} - \sum_{r,s<0}\Big)(-1)^{r+s}x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}. \tag{2.1}$$

Next, consider the Appell-Lerch series

$$m(x,q,z) := \frac{1}{j(z;q)}\sum_{r\in\mathbb{Z}}\frac{(-1)^r q^{\binom{r}{2}} z^r}{1-q^{r-1}xz}, \tag{2.2}$$

where $x,\ z \in \mathbb{C}^* := \mathbb{C}\setminus\{0\}$ with neither z nor xz an integral power of q in order to avoid poles. One of the main results in [14] expresses (2.1) in terms of (2.2). Let

$$\begin{aligned} g_{a,b,c}(x,y,q,z_1,z_0) :=& \sum_{t=0}^{a-1}(-y)^t q^{c\binom{t}{2}} j(q^{bt}x;q^a) m\Big(-q^{a\binom{b+1}{2}-c\binom{a+1}{2}-t(b^2-ac)}\frac{(-y)^a}{(-x)^b}, q^{a(b^2-ac)}, z_0\Big) \\ &+ \sum_{t=0}^{c-1}(-x)^t q^{a\binom{t}{2}} j(q^{bt}y;q^c) m\Big(-q^{c\binom{b+1}{2}-a\binom{c+1}{2}-t(b^2-ac)}\frac{(-x)^c}{(-y)^b}, q^{c(b^2-ac)}, z_1\Big). \end{aligned} \tag{2.3}$$

Following [14], we use the term "generic" to mean that the parameters do not cause poles in the Appell-Lerch sums or in the quotients of theta functions.

Theorem 2.1 ([14], Theorem 1.3)**.** *Let n and p be positive integers with $(n,p)=1$. For generic $x,\ y\in\mathbb{C}^*$,*

$$f_{n,n+p,n}(x,y,q) = g_{n,n+p,n}(x,y,q,-1,-1) + \frac{1}{\overline{J}_{0,np(2n+p)}}\theta_{n,p}(x,y,q),$$

where

$$\theta_{n,p}(x,y,q) := \sum_{r^*=0}^{p-1}\sum_{s^*=0}^{p-1} q^{n\binom{r-(n-1)/2}{2}+(n+p)(r-(n-1)/2)(s+(n+1)/2)+n\binom{s+(n+1)/2}{2}}(-x)^{r-(n-1)/2}$$
$$\times \frac{(-y)^{s+(n+1)/2}J^3_{p^2(2n+p)}j(-q^{np(s-r)}\frac{x^n}{y^n};q^{np^2})j(q^{p(2n+p)(r+s)+p(n+p)}(xy)^p;q^{p^2(2n+p)})}{j(q^{p(2n+p)r+p(n+p)/2}\frac{(-y)^{n+p}}{(-x)^n}, q^{p(2n+p)s+p(n+p)/2}\frac{(-x)^{n+p}}{(-y)^n};q^{p^2(2n+p)})}.$$

Here, $r := r^* + \{(n-1)/2\}$ *and* $s := s^* + \{(n-1)/2\}$ *with* $0 \leq \{\alpha\} < 1$ *denoting the fractional part of* α.

We can now prove Theorem 1.2.

Proof of Theorem 1.2. The first step is to recognize $B_N(q)$ in a different context. For $N \geq 1$, consider the string function of level N of the affine Lie algebra $A_1^{(1)}$ (e.g., see [12, 15])

$$\mathcal{C}^N_{m,\ell}(q) = \frac{q^{\frac{m^2-\ell^2}{4N}}}{(q)_\infty} \sum_{\substack{\mathbf{n}\in\mathbb{Z}^{N-1}_{\geq 0} \\ \frac{m+\ell}{2N}+(C^{-1}\mathbf{n})_1\in\mathbb{Z}}} \frac{q^{\mathbf{n}C^{-1}(\mathbf{n}-\mathbf{e}_\ell)^T}}{(q)_{n_1}\cdots(q)_{n_{N-1}}}, \tag{2.4}$$

where $\mathbf{n} = (n_1, \dots, n_{N-1})$, $\mathbf{e}_i$ is the i-th standard unit vector in $\mathbb{Z}^{N-1}$ (with $\mathbf{e}_0 = \mathbf{e}_N = 0$), C is the A_{N-1} Cartan matrix whose inverse C^{-1} is given by

$$(C^{-1})_{i,j} = \min(i,j) - \frac{ij}{N},$$

and $(C^{-1}\mathbf{n})_1$ is the first entry in the vector $C^{-1}\mathbf{n}$. A straightforward computation (see the proof of Theorem 6 in [2]) yields

$$B_N(q) = \sum_{\substack{\mathbf{n}\in\mathbb{Z}^{N-1}_{\geq 0} \\ (C^{-1}\mathbf{n})_1\in\mathbb{Z}}} \frac{q^{\mathbf{n}C^{-1}\mathbf{n}^T}}{(q)_{n_1}\cdots(q)_{n_{N-1}}}. \tag{2.5}$$

Comparing (2.4) when $\ell = 0$ and m is divisible by $2N$ with (2.5), we have for all $N \geq 2$,

$$B_N(q) = q^{\frac{-m^2}{4N}}(q)_\infty \mathcal{C}^N_{m,0}(q). \tag{2.6}$$

Next, by Example 1.3 on page 386 of [14], we have

$$\mathcal{C}^N_{m,0}(q) = \frac{1}{(q)^3_\infty} f_{1,N+1,1}(q^{1+m/2}, q^{1-m/2}, q).$$

Thus from (2.6), we deduce that for all $N \geq 2$ and m divisible by $2N$,

$$B_N(q) = \frac{q^{\frac{-m^2}{4N}}}{(q)^2_\infty} f_{1,N+1,1}(q^{1+m/2}, q^{1-m/2}, q). \tag{2.7}$$

By Theorem 2.1, we have

$$f_{1,N+1,1}(q^{1+m/2}, q^{1-m/2}, q) = g_{1,N+1,1}(q^{1+m/2}, q^{1-m/2}, q, -1, -1) + \frac{1}{\overline{J}_{0,N(N+2)}}\theta_{1,N}(q^{1+m/2}, q^{1-m/2}, q).$$

Now, observe that

$$g_{1,N+1,1}(q^{1+m/2}, q^{1-m/2}, q, -1, -1) = 0$$

as there are no poles in the Appell-Lerch series

$$m(q^{N(N+1)/2+m(N+2)/2}, q^{N(N+2)}, -1)$$

and

$$m(q^{N(N+1)/2-m(N+2)/2}, q^{N(N+2)}, -1)$$

(indeed, this is true whenever $m(N+2)/2 \not\equiv \pm N(N+1)/2 \pmod{N(N+2)}$, which is always the case when $m \equiv 0 \pmod{2N}$) and $j(q^{1+m/2}; q) = j(q^{1-m/2}; q) = 0$. Thus,

$$B_N(q) = \frac{q^{\frac{-m^2}{4N}}}{(q)_\infty^2 \overline{J}_{0,N(N+2)}}\theta_{1,N}(q^{1+m/2}, q^{1-m/2}, q).$$

We now take $m = 0$. The factor $q^{\frac{-m^2}{4N}}$ disappears and $\theta_{1,N}(q, q, q)$ is given as in (1.4). This proves the result.

$\square$

3. Concluding remarks

There are several avenues for further study. First, Table 1 suggests that as $N \to \infty$, the limiting value of $B_N(q)$ is strictly larger than $(q)_\infty$. This contradicts [2, Corollary 11, (2)]. Thus, it would be desirable to compute both asymptotics for $B_N(q)$ and the correct limiting value of $\psi(n, k, q, N)$ as $N \to \infty$. Second, for $N = 2$, 3 and 4, one can reduce the number of products of quotients of theta functions occurring in Theorem 1.2 by first invoking Theorems 1.9–1.11 in [14], then performing routine (yet possibly involved) simplifications [6]. In these cases, we require that $m \equiv 0 \pmod{2N}$, $m \not\equiv 0 \pmod{N(N+2)}$ and, if m is odd, $m \not\equiv \pm(N+1) \pmod{2(N+2)}$. For example, one can recover (1.3) in this manner. The details are left to the interested reader. Third, given that (2.6) is a key step in the proof of Theorem 1.2, it is natural to wonder if string functions which generalize (2.4) (see [9, 11]) can also be realized in terms of computing an appropriate probability. Finally, can Theorem 1.2 be understood via Markov chains, group theory or, possibly, Hall-Littlewood functions [16]?

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