Review # 1 - Key

1) a) (i) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 3-x = 3$

(ii) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \sqrt{-x} = 0$

(iii) By (i) and (ii), $\lim_{x\to 0} f(x)$ DNE

(iv) $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 3-x = 0$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (x-3)^2 = 0$

$\Rightarrow$ Thus $\lim_{x\to 3} f(x) = 0$

b) f is discontinuous at $x=0$ since $\lim_{x\to 0} f(x)$ DNE

f is discontinuous at $x=3$ since $f(3)$ is undefined.

2) a)
$$\lim_{x\to 2} \frac{\sqrt{x+2}-\sqrt{2x}}{x^2-2x}\cdot\frac{\sqrt{x+2}+\sqrt{2x}}{\sqrt{x+2}+\sqrt{2x}} = \lim_{x\to 2}\frac{x+2-2x}{(x^2-2x)(\sqrt{x+2}+\sqrt{2x})}$$
$$= \lim_{x\to 2}\frac{\overset{-1}{\cancel{(2-x)}}}{x\cancel{(x-2)}(\sqrt{x+2}+\sqrt{2x})}$$
$$= \lim_{x\to 2}\frac{-1}{x(\sqrt{x+2}+\sqrt{2x})} = -\frac{1}{8}$$

b)
$$\lim_{x\to\infty}\frac{x^4-3x}{x^4+100x^3+x^2+x+7} \underset{\text{Divide by } x^4}{=} \lim_{x\to\infty}\frac{\frac{x^4}{x^4}-\frac{3x}{x^4}}{\frac{x^4}{x^4}+\frac{100x^3}{x^4}+\frac{x^2}{x^4}+\frac{x}{x^4}+\frac{7}{x^4}} = \lim_{x\to\infty}\frac{1-\overset{0}{\cancel{\frac{3}{x^3}}}}{1+\underset{0}{\cancel{\frac{100}{x}}}+\underset{0}{\cancel{\frac{1}{x^2}}}+\underset{0}{\cancel{\frac{1}{x^3}}}+\underset{0}{\cancel{\frac{7}{x^4}}}} = 1$$

3) A function f is continuous at a if:

i) $f(a)$ exists

ii) $\lim_{x\to a} f(x)$ exists

iii) $\lim_{x\to a} f(x) = f(a)$

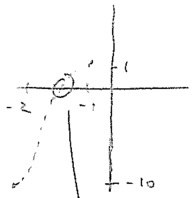
4) Use the Squeeze Thm: i.e. Since $3x \le f(x) \le x^3+2$,

$$\lim_{x\to 1} 3x \le \lim_{x\to 1} f(x) \le \lim_{x\to 1} x^3+2$$

$\Rightarrow 3 \le \lim_{x\to 1} f(x) \le 3 \Rightarrow$ By the Squeeze Thm, $\lim_{x\to 1} f(x) = 3$.

5) Set $f(x) = 2x^3 + x^2 + 2$. So we're looking for a c in $(-2,-1)$ such that $f(c)=0$. As $f(x)$ is a polynomial, it is continuous everywhere (i.e. $(-\infty,\infty)$) and hence on $(-2,-1)$.

$$\left.\begin{array}{l} \text{At } x=-2 \Rightarrow f(-2) = 2(-2)^3 + (-2)^2 + 2 = -10 \\ \text{At } x=-1 \Rightarrow f(-1) = 2(-1)^3 + (-1)^2 + 2 = 1 \end{array}\right\} \Rightarrow$$



By the Intermediate Value Thm, there exists a c in $(-2,-1)$ s.t. $f(c)=0$ (i.e. f crosses x-axis)

6) $f(x) = \dfrac{4-x}{x+3}$

V.A. Check that $\displaystyle\lim_{x\to -3^+} \frac{4-x}{x+3} = \pm\infty$

$$\lim_{x\to -3^+} \frac{4-x}{x+3} = \left[\frac{7}{\text{small positive}}\right] \to \infty \checkmark$$ So V.A. at $x=-3$

H.A. $$\lim_{x\to\infty} \frac{4-x}{x+3} = \lim_{x\to\infty} \frac{\frac{4}{x}-1}{1+\frac{3}{x}} = -1$$, So H.A. at $y=-1$

7) Slope of tangent line at $x=1$ $$= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^2+2(1+h)+1-4}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{1}+2h+h^2+\cancel{2}+2h+\cancel{1}-\cancel{4}}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{h}(4+h)}{\cancel{h}} = \lim_{h\to 0} 4+h = 4$$

Now, use point-slope form $\Rightarrow$ $y-y_1 = m(x-x_1)$
$y-4 = 4(x-1) \Rightarrow y = 4x$

8) Given $f(x) = \sqrt{3-5x}$, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{3-5(x+h)} - \sqrt{3-5x}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3-5x-5h} - \sqrt{3-5x}}{h} \cdot \frac{\sqrt{3-5x-5h} + \sqrt{3-5x}}{\sqrt{3-5x-5h} + \sqrt{3-5x}}$$

$$= \lim_{h\to 0} \frac{3-5x-5h-\cancel{3}+\cancel{5x}}{h\left(\sqrt{3-5x-5h}+\sqrt{3-5x}\right)} = \lim_{h\to 0} \frac{-5}{\sqrt{3-5x-5h}+\sqrt{3-5x}}$$

$$= \frac{-5}{2\sqrt{3-5x}}.$$

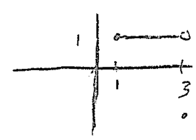
9) (i) False. Consider $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x^4}$. Then $\lim_{x\to 0} \frac{1}{x^2} = \infty$ and $\lim_{x\to 0} \frac{1}{x^4} = \infty$,

but $\lim_{x\to 0} \frac{1}{x^2} - \frac{1}{x^4} = -\infty$.

(ii) False. Consider $f(x) = |x|$ at 0. Continuous at 0, but not differentiable at 0.

(iii) False. Recall the function must be continuous on $[1,3]$ in order to find a c in $(1,3)$ such that $f(c) = 0$. For example, let

$$f(x) = \begin{cases} 1 & \text{if } 1 \le x < 3 \\ -1 & \text{if } x = 3 \end{cases} \implies$$



Then there is no c in $[1,3]$ s.t. $f(c) = 0$

(iv) True. A polynomial is continuous everywhere (i.e. $(-\infty,\infty)$) and so we can calculate the limit by substitution i.e. $\lim_{x\to a} p(x) = p(a)$.

10) Given $H(x) = (x^3 - x + 1)(x^{-2} + 2x^{-3}) + \dfrac{x}{\sin x + \cos x}$.

Use the product rule and the quotient rule.

Then $H'(x) = (x^3 - x + 1)\dfrac{d}{dx}(x^{-2} + 2x^{-3}) + (x^{-2} + 2x^{-3})\dfrac{d}{dx}(x^3 - x + 1)$

$$+ \frac{(\sin x + \cos x)\frac{d}{dx}(x) - x\frac{d}{dx}(\sin x + \cos x)}{(\sin x + \cos x)^2}$$

$$= (x^3 - x + 1)(-2x^{-3} - 6x^{-4}) + (x^{-2} + 2x^{-3})(3x^2 - 1)$$

$$+ \frac{(\sin x + \cos x)(1) - x(\cos x - \sin x)}{(\sin x + \cos x)^2}$$

$$= -2 - 6x^{-1} + 2x^{-2} + 6x^{-3} - 2x^{-3} - 6x^{-4} + 3 - x^{-2} + 6x^{-1} - 2x^{-3}$$

$$+ \frac{\sin x + \cos x - x\cos x + x\sin x}{(\sin x + \cos x)^2}$$

$$= 1 + x^{-2} + 2x^{-3} - 6x^{-4}$$

$$+ \frac{\sin x + \cos x - x\cos x + x\sin x}{(\sin x + \cos x)^2}.$$