MST 10010: Calculus I

Exercise Set 5

Do only the following problems:

Section 4.1 (pages 284-285): 5, 7, 9, 13, 15, 17, 19, 21, 23, 25, 31, 33, 35, 37, 41, 43, 45, 47, 49, 51, 53, 55, 57, 59, 61, 63.

Section 4.2 (pages 289-289): 1, 3, 5, 7, 9, 13, 15, 17, 23, 25

Section 4.3 (pages 302-304): 3, 5, 7, 11, 13, 15, 17, 19, 21 and 23 (just use the First Derivative Test), 25, 27, 29 (a)-(d), 31, 33, 35, 37, 39 (a)-(c), 43, 45 (a)-(d).

Section 4.5 (pages 321-322): 1, 3, 5, 7, 9, 11, 13, 15, 27, 45.

Section 4.4 (pages 311-312): 5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45, 49, 53, 57, 61.

Section 4.9 (page 349): 5, 7, 9, 11, 13.

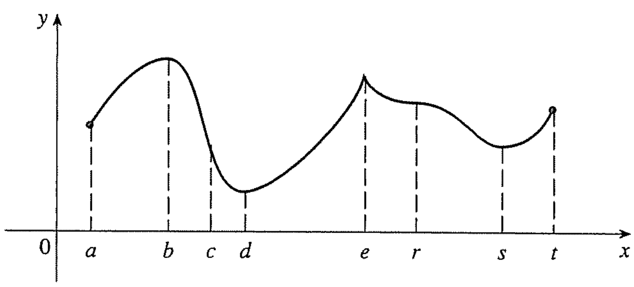
4.1 Exercises

1. Explain the difference between an absolute minimum and a local minimum.

2. Suppose f is a continuous function defined on a closed interval $[a, b]$.
   (a) What theorem guarantees the existence of an absolute maximum value and an absolute minimum value for f?
   (b) What steps would you take to find those maximum and minimum values?

3–4 □ For each of the numbers a, b, c, d, e, r, s, and t, state whether the function whose graph is shown has an absolute maximum or minimum, a local maximum or minimum, or neither a maximum nor a minimum.

3.



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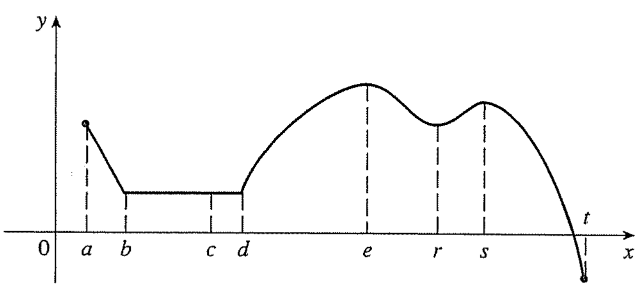
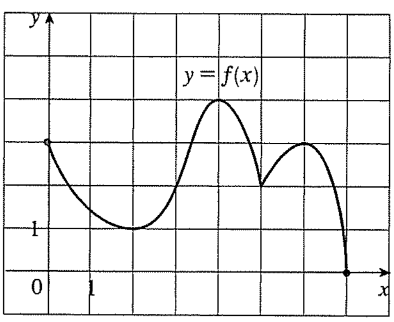


5–6 □ Use the graph to state the absolute and local maximum and minimum values of the function.

5.



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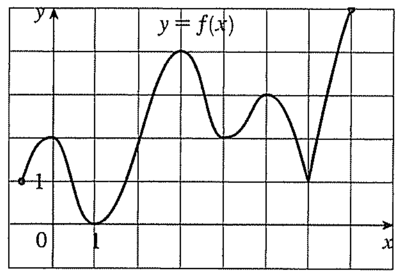


7–10 □ Sketch the graph of a function f that is continuous on $[0, 3]$ and has the given properties.

7. Absolute maximum at 0, absolute minimum at 3, local minimum at 1, local maximum at 2

8. Absolute maximum at 1, absolute minimum at 2

9. 2 is a critical number, but f has no local maximum or minimum

10. Absolute minimum at 0, absolute maximum at 2, local maxima at 1 and 2, local minimum at 1.5

11. (a) Sketch the graph of a function that has a local maximum at 2 and is differentiable at 2.
    (b) Sketch the graph of a function that has a local maximum at 2 and is continuous but not differentiable at 2.
    (c) Sketch the graph of a function that has a local maximum at 2 and is not continuous at 2.

12. (a) Sketch the graph of a function on $[-1, 2]$ that has an absolute maximum but no local maximum.
    (b) Sketch the graph of a function on $[-1, 2]$ that has a local maximum but no absolute maximum.

13. (a) Sketch the graph of a function on $[-1, 2]$ that has an absolute maximum but no absolute minimum.
    (b) Sketch the graph of a function on $[-1, 2]$ that is discontinuous but has both an absolute maximum and an absolute minimum.

14. (a) Sketch the graph of a function that has two local maxima, one local minimum, and no absolute minimum.
    (b) Sketch the graph of a function that has three local minima, two local maxima, and seven critical numbers.

15–30 □ Find the absolute and local maximum and minimum values of f. Begin by sketching its graph by hand. (Use the graphs and transformations of Sections 1.2 and 1.3.)

15. $f(x) = 8 - 3x, \quad x \geqslant 1$

16. $f(x) = 3 - 2x, \quad x \leqslant 5$

17. $f(x) = x^2, \quad 0 < x < 2$

18. $f(x) = x^2, \quad 0 < x \leqslant 2$

19. $f(x) = x^2, \quad 0 \leqslant x < 2$

20. $f(x) = x^2, \quad 0 \leqslant x \leqslant 2$

21. $f(x) = x^2, \quad -3 \leqslant x \leqslant 2$

22. $f(x) = 1 + (x + 1)^2, \quad -2 \leqslant x < 5$

23. $f(t) = 1/t, \quad 0 < t < 1$

24. $f(t) = 1/t, \quad 0 < t \leqslant 1$

25. $f(\theta) = \sin \theta, \quad -2\pi \leqslant \theta \leqslant 2\pi$

26. $f(\theta) = \tan \theta, \quad -\pi/4 \leqslant \theta < \pi/2$

27. $f(x) = x^5$

28. $f(x) = 2 - x^4$

29. $f(x) = 1 - e^{-x}, \quad x \geqslant 0$

30. $f(x) = \begin{cases} x^2 & \text{if } -1 \leqslant x < 0 \\ 2 - x^2 & \text{if } 0 \leqslant x \leqslant 1 \end{cases}$

31–48 □ Find the critical numbers of the function.

31. $f(x) = 5x^2 + 4x$

32. $f(x) = 5 + 6x - 2x^3$

33. $f(t) = 2t^3 + 3t^2 + 6t + 4$

34. $f(x) = 4x^3 - 9x^2 - 12x + 3$

35. $s(t) = 2t^3 + 3t^2 - 6t + 4$

36. $s(t) = t^4 + 4t^3 + 2t^2$

37. $f(r) = \dfrac{r}{r^2 + 1}$

38. $f(z) = \dfrac{z + 1}{z^2 + z + 1}$

39. $g(x) = |2x + 3|$

40. $g(x) = x^{1/3} - x^{-2/3}$

41. $g(t) = 5t^{2/3} + t^{5/3}$

42. $g(t) = \sqrt{t}\,(1 - t)$

43. $F(x) = x^{4/5}(x - 4)^2$

44. $G(x) = \sqrt[3]{x^2 - x}$

45. $f(\theta) = \sin^2(2\theta)$

46. $g(\theta) = \theta + \sin \theta$

47. $f(x) = x \ln x$

48. $f(x) = xe^{2x}$

49–64 □ Find the absolute maximum and absolute minimum values of f on the given interval.

49. $f(x) = 3x^2 - 12x + 5, \quad [0, 3]$

50. $f(x) = x^3 - 3x + 1, \quad [0, 3]$

51. $f(x) = 2x^3 + 3x^2 + 4, \quad [-2, 1]$

52. $f(x) = 18x + 15x^2 - 4x^3, \quad [-3, 4]$

53. $f(x) = x^4 - 4x^2 + 2, \quad [-3, 2]$

54. $f(x) = 3x^5 - 5x^3 - 1, \quad [-2, 2]$

55. $f(x) = x^2 + 2/x, \quad [\frac{1}{2}, 2]$

56. $f(x) = \sqrt{9 - x^2}, \quad [-1, 2]$

57. $f(x) = \dfrac{x}{x^2 + 1}, \quad [0, 2]$

58. $f(x) = \dfrac{x}{x + 1}, \quad [1, 2]$

59. $f(x) = \sin x + \cos x, \quad [0, \pi/3]$

60. $f(x) = x - 2\cos x, \quad [-\pi, \pi]$

61. $f(x) = xe^{-x}, \quad [0, 2]$

62. $f(x) = (\ln x)/x, \quad [1, 3]$

63. $f(x) = x - 3\ln x, \quad [1, 4]$

64. $f(x) = e^{-x} - e^{-2x}, \quad [0, 1]$

65–66 □ Use a graph to estimate the critical numbers of f to one decimal place.

65. $f(x) = x^4 - 3x^2 + x$

66. $f(x) = |x^3 - 3x^2 + 2|$

67–70 □

(a) Use a graph to estimate the absolute maximum and minimum values of the function to two decimal places.

(b) Use calculus to find the exact maximum and minimum values.

67. $f(x) = x^3 - 8x + 1, \quad -3 \leqslant x \leqslant 3$

68. $f(x) = e^{x^3 - x}, \quad -1 \leqslant x \leqslant 0$

69. $f(x) = x\sqrt{x - x^2}$

70. $f(x) = (\cos x)/(2 + \sin x), \quad 0 \leqslant x \leqslant 2\pi$

71. Between 0 °C and 30 °C, the volume V (in cubic centimeters) of 1 kg of water at a temperature T is given approximately by the formula

$$V = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3$$

Find the temperature at which water has its maximum density.

72. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin \theta + \cos \theta}$$

where μ is a positive constant called the *coefficient of friction* and where $0 \leqslant \theta \leqslant \pi/2$. Show that F is minimized when $\tan \theta = \mu$.

73. A model for the food-price index (the price of a representative "basket" of foods) between 1984 and 1994 is given by the function

$$I(t) = 0.00009045t^5 + 0.001438t^4 - 0.06561t^3 + 0.4598t^2 - 0.6270t + 99.33$$

where t is measured in years since midyear 1984, so $0 \leqslant t \leqslant 10$, and $I(t)$ is measured in 1987 dollars and scaled such that $I(3) = 100$. Estimate the times when food was cheapest and most expensive during the period 1984–1994.

74. On May 7, 1992, the space shuttle *Endeavour* was launched on mission STS-49, the purpose of which was to install a new perigee kick motor in an Intelsat communications satellite. The

Since $f'(x) = 0$ for all x, we have $f'(c) = 0$, and so Equation 6 becomes

$$f(x_2) - f(x_1) = 0 \qquad \text{or} \qquad f(x_2) = f(x_1)$$

Therefore f has the same value at *any* two numbers x_1 and x_2 in (a, b). This means that f is constant on (a, b).

7 Corollary If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

PROOF Let $F(x) = f(x) - g(x)$. Then

$$F'(x) = f'(x) - g'(x) = 0$$

for all x in (a, b). Thus, by Theorem 5, F is constant; that is, $f - g$ is constant.

NOTE Care must be taken in applying Theorem 5. Let

$$f(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

The domain of f is $D = \{x \mid x \neq 0\}$ and $f'(x) = 0$ for all x in D. But f is obviously not a constant function. This does not contradict Theorem 5 because D is not an interval. Notice that f is constant on the interval $(0, \infty)$ and also on the interval $(-\infty, 0)$.

EXAMPLE 6 Prove the identity $\tan^{-1}x + \cot^{-1}x = \pi/2$.

SOLUTION Although calculus isn't needed to prove this identity, the proof using calculus is quite simple. If $f(x) = \tan^{-1}x + \cot^{-1}x$, then

$$f'(x) = \frac{1}{1 + x^2} - \frac{1}{1 + x^2} = 0$$

for all values of x. Therefore $f(x) = C$, a constant. To determine the value of C, we put $x = 1$ [because we can evaluate $f(1)$ exactly]. Then

$$C = f(1) = \tan^{-1}1 + \cot^{-1}1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

Thus $\tan^{-1}x + \cot^{-1}x = \pi/2$.

4.2 Exercises

1–4 Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

1. $f(x) = 5 - 12x + 3x^2$, $[1, 3]$

2. $f(x) = x^3 - x^2 - 6x + 2$, $[0, 3]$

3. $f(x) = \sqrt{x} - \frac{1}{3}x$, $[0, 9]$

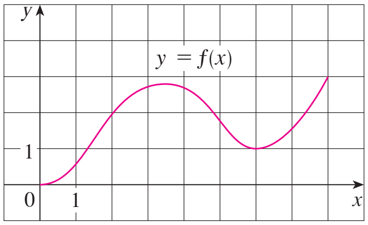
4. $f(x) = \cos 2x$, $[\pi/8, 7\pi/8]$

5. Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1, 1)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

6. Let $f(x) = \tan x$. Show that $f(0) = f(\pi)$ but there is no number c in $(0, \pi)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

Graphing calculator or computer required   **1.** Homework Hints available at stewartcalculus.com

7. Use the graph of f to estimate the values of c that satisfy the conclusion of the Mean Value Theorem for the interval $[0, 8]$.



8. Use the graph of f given in Exercise 7 to estimate the values of c that satisfy the conclusion of the Mean Value Theorem for the interval $[1, 7]$.

9–12 Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

9. $f(x) = 2x^2 - 3x + 1, \quad [0, 2]$

10. $f(x) = x^3 - 3x + 2, \quad [-2, 2]$

11. $f(x) = \ln x, \quad [1, 4]$

12. $f(x) = 1/x, \quad [1, 3]$

13–14 Find the number c that satisfies the conclusion of the Mean Value Theorem on the given interval. Graph the function, the secant line through the endpoints, and the tangent line at $(c, f(c))$. Are the secant line and the tangent line parallel?

13. $f(x) = \sqrt{x}, \quad [0, 4]$

14. $f(x) = e^{-x}, \quad [0, 2]$

15. Let $f(x) = (x - 3)^{-2}$. Show that there is no value of c in $(1, 4)$ such that $f(4) - f(1) = f'(c)(4 - 1)$. Why does this not contradict the Mean Value Theorem?

16. Let $f(x) = 2 - |2x - 1|$. Show that there is no value of c such that $f(3) - f(0) = f'(c)(3 - 0)$. Why does this not contradict the Mean Value Theorem?

17–18 Show that the equation has exactly one real root.

17. $2x + \cos x = 0$

18. $x^3 + e^x = 0$

19. Show that the equation $x^3 - 15x + c = 0$ has at most one root in the interval $[-2, 2]$.

20. Show that the equation $x^4 + 4x + c = 0$ has at most two real roots.

21. (a) Show that a polynomial of degree 3 has at most three real roots.
(b) Show that a polynomial of degree n has at most n real roots.

22. (a) Suppose that f is differentiable on $\mathbb{R}$ and has two roots. Show that f' has at least one root.
(b) Suppose f is twice differentiable on $\mathbb{R}$ and has three roots. Show that f'' has at least one real root.
(c) Can you generalize parts (a) and (b)?

23. If $f(1) = 10$ and $f'(x) \geqslant 2$ for $1 \leqslant x \leqslant 4$, how small can $f(4)$ possibly be?

24. Suppose that $3 \leqslant f'(x) \leqslant 5$ for all values of x. Show that $18 \leqslant f(8) - f(2) \leqslant 30$.

25. Does there exist a function f such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leqslant 2$ for all x?

26. Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b). Suppose also that $f(a) = g(a)$ and $f'(x) < g'(x)$ for $a < x < b$. Prove that $f(b) < g(b)$. [*Hint:* Apply the Mean Value Theorem to the function $h = f - g$.]

27. Show that $\sqrt{1 + x} < 1 + \frac{1}{2}x$ if $x > 0$.

28. Suppose f is an odd function and is differentiable everywhere. Prove that for every positive number b, there exists a number c in $(-b, b)$ such that $f'(c) = f(b)/b$.

29. Use the Mean Value Theorem to prove the inequality

$$|\sin a - \sin b| \leqslant |a - b| \qquad \text{for all } a \text{ and } b$$

30. If $f'(x) = c$ (c a constant) for all x, use Corollary 7 to show that $f(x) = cx + d$ for some constant d.

31. Let $f(x) = 1/x$ and

$$g(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\ 1 + \dfrac{1}{x} & \text{if } x < 0 \end{cases}$$

Show that $f'(x) = g'(x)$ for all x in their domains. Can we conclude from Corollary 7 that $f - g$ is constant?

32. Use the method of Example 6 to prove the identity

$$2\sin^{-1}x = \cos^{-1}(1 - 2x^2) \qquad x \geqslant 0$$

33. Prove the identity

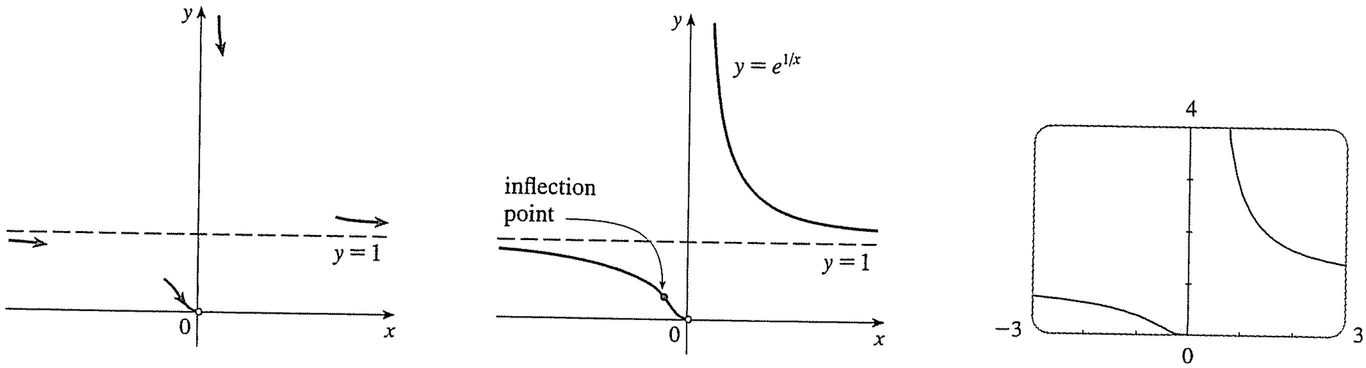
$$\arcsin\frac{x - 1}{x + 1} = 2\arctan\sqrt{x} - \frac{\pi}{2}$$

34. At 2:00 PM a car's speedometer reads 30 mi/h. At 2:10 PM it reads 50 mi/h. Show that at some time between 2:00 and 2:10 the acceleration is exactly 120 mi/h^2.

35. Two runners start a race at the same time and finish in a tie. Prove that at some time during the race they have the same speed. [*Hint:* Consider $f(t) = g(t) - h(t)$, where g and h are the position functions of the two runners.]

36. A number a is called a **fixed point** of a function f if $f(a) = a$. Prove that if $f'(x) \neq 1$ for all real numbers x, then f has at most one fixed point.

To sketch the graph of f we first draw the horizontal asymptote $y = 1$ (as a dashed line), together with the parts of the curve near the asymptotes in a preliminary sketch [Figure 13(a)]. These parts reflect the information concerning limits and the fact that f is decreasing on both $(-\infty, 0)$ and $(0, \infty)$. Notice that we have indicated that $f(x) \to 0$ as $x \to 0^-$ even though $f(0)$ does not exist. In Figure 13(b) we finish the sketch by incorporating the information concerning concavity and the inflection point. In Figure 13(c) we check our work with a graphing device.



(a) Preliminary sketch (b) Finished sketch (c) Computer confirmation

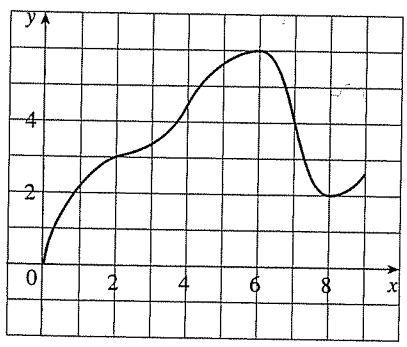
FIGURE 13

4.3 Exercises

1–2 □ Use the given graph of f to find the following.

(a) The largest open intervals on which f is increasing.
(b) The largest open intervals on which f is decreasing.
(c) The largest open intervals on which f is concave upward.
(d) The largest open intervals on which f is concave downward.
(e) The coordinates of the points of inflection.

1.



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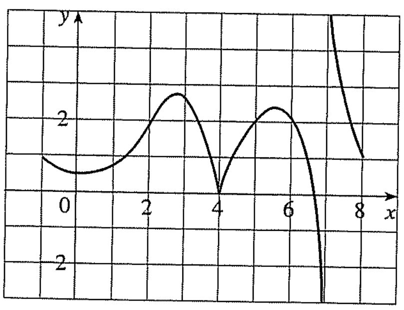
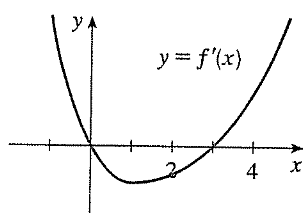


3. Suppose you are given a formula for a function f.
(a) How do you determine where f is increasing or decreasing?
(b) How do you determine where the graph of f is concave upward or concave downward?
(c) How do you locate inflection points?

4. (a) State the First Derivative Test.
(b) State the Second Derivative Test. Under what circumstances is it inconclusive? What do you do if it fails?

5–6 □ The graph of the *derivative* f' of a function f is shown.
(a) On what intervals is f increasing or decreasing?
(b) At what values of x does f have a local maximum or minimum?

5.



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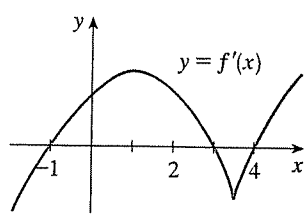
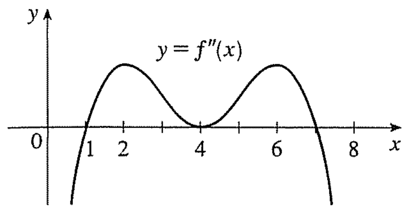
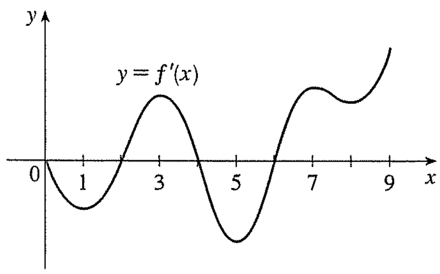


7. The graph of the second derivative f'' of a function f is shown. State the x-coordinates of the inflection points of f. Give reasons for your answers.

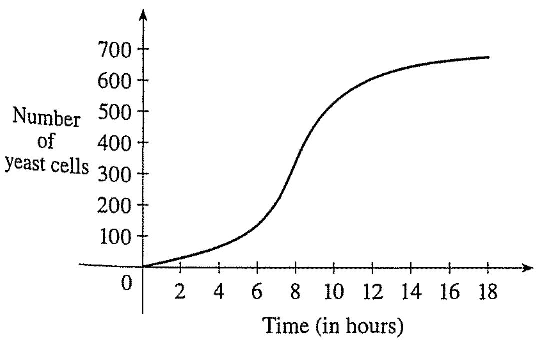


8. The graph of the first derivative f' of a function f is shown.
(a) On what intervals is f increasing? Explain.
(b) At what values of x does f have a local maximum or minimum? Explain.
(c) On what intervals is f concave upward or concave downward? Explain.
(d) What are the x-coordinates of the inflection points of f? Why?



9. Sketch the graph of a function whose first and second derivatives are always negative.

10. A graph of a population of yeast cells in a new laboratory culture as a function of time is shown.
(a) Describe how the rate of population increase varies.
(b) When is this rate highest?
(c) On what intervals is the population function concave upward or downward?
(d) Estimate the coordinates of the inflection point.



11–20 □
(a) Find the intervals on which f is increasing or decreasing.
(b) Find the local maximum and minimum values of f.
(c) Find the intervals of concavity and the inflection points.

Resources / Module 3 / Concavity / Problems and Tests

11. $f(x) = x^3 - 12x + 1$

12. $f(x) = 5 - 3x^2 + x^3$

13. $f(x) = x^6 + 192x + 17$

14. $f(x) = \dfrac{x}{(1 + x)^2}$

15. $f(x) = x - 2\sin x, \quad 0 < x < 3\pi$

16. $f(x) = 2\sin x + \sin^2 x, \quad 0 \leqslant x \leqslant 2\pi$

17. $f(x) = xe^x$

18. $f(x) = x^2 e^x$

19. $f(x) = (\ln x)/\sqrt{x}$

20. $f(x) = x \ln x$

21–23 □ Find the local maximum and minimum values of f using both the First and Second Derivative Tests. Which method do you prefer?

21. $f(x) = x^5 - 5x + 3$

22. $f(x) = \dfrac{x}{x^2 + 4}$

23. $f(x) = x + \sqrt{1 - x}$

24. (a) Find the critical numbers of $f(x) = x^4(x - 1)^3$.
(b) What does the Second Derivative Test tell you about the behavior of f at these critical numbers?
(c) What does the First Derivative Test tell you?

25–28 □ Sketch the graph of a function that satisfies all of the given conditions.

25. $f'(-1) = f'(1) = 0, \quad f'(x) < 0$ if $|x| < 1$,
$f'(x) > 0$ if $|x| > 1, \quad f(-1) = 4, \quad f(1) = 0$,
$f''(x) < 0$ if $x < 0, \quad f''(x) > 0$ if $x > 0$

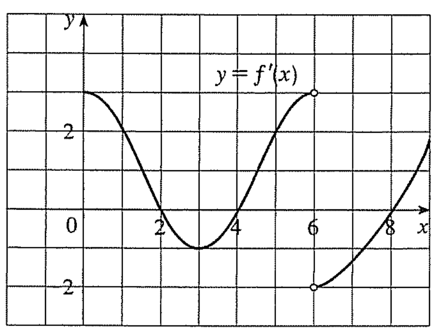
26. $f'(-1) = 0, \quad f'(1)$ does not exist,
$f'(x) < 0$ if $|x| < 1, \quad f'(x) > 0$ if $|x| > 1$,
$f(-1) = 4, \quad f(1) = 0, \quad f''(x) < 0$ if $x \neq 1$

27. $f'(2) = 0, \quad f(2) = -1, \quad f(0) = 0$,
$f'(x) < 0$ if $0 < x < 2, \quad f'(x) > 0$ if $x > 2$,
$f''(x) < 0$ if $0 \leqslant x < 1$ or if $x > 4$,
$f''(x) > 0$ if $1 < x < 4, \quad \lim_{x\to\infty} f(x) = 1$,
$f(-x) = f(x)$ for all x

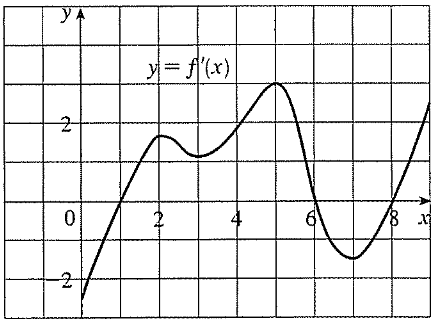
28. $\lim_{x\to 3} f(x) = -\infty, \quad f''(x) < 0$ if $x \neq 3, \quad f'(0) = 0$,
$f'(x) > 0$ if $x < 0$ or $x > 3, \quad f'(x) < 0$ if $0 < x < 3$

29–30 □ The graph of the derivative f' of a continuous function f is shown.
(a) On what intervals is f increasing or decreasing?
(b) At what values of x does f have a local maximum or minimum?
(c) On what intervals is f concave upward or downward?
(d) State the x-coordinates of the points of inflection.
(e) Assuming that $f(0) = 0$, sketch a graph of f.

29.



30.



31–42 □

(a) Find the intervals of increase or decrease.
(b) Find the local maximum and minimum values.
(c) Find the intervals of concavity and the inflection points.
(d) Use the information from parts (a), (b), and (c) to sketch the graph. Check your work with a graphing device if you have one.

31. $f(x) = 2x^3 - 3x^2 - 12x$

32. $f(x) = 2 + 3x - x^3$

33. $f(x) = x^4 - 6x^2$

34. $g(x) = 200 + 8x^3 + x^4$

35. $h(x) = 3x^5 - 5x^3 + 3$

36. $h(x) = (x^2 - 1)^3$

37. $P(x) = x\sqrt{x^2 + 1}$

38. $Q(x) = x - 3x^{1/3}$

39. $Q(x) = x^{1/3}(x + 3)^{2/3}$

40. $f(x) = \ln(1 + x^2)$

41. $f(\theta) = \sin^2\theta, \quad 0 \leqslant \theta \leqslant 2\pi$

42. $f(t) = t + \cos t, \quad -2\pi \leqslant \theta \leqslant 2\pi$

43–50 □

(a) Find the vertical and horizontal asymptotes.
(b) Find the intervals of increase or decrease.
(c) Find the local maximum and minimum values.
(d) Find the intervals of concavity and the inflection points.
(e) Use the information from parts (a)–(d) to sketch the graph of f.

43. $f(x) = \dfrac{1 + x^2}{1 - x^2}$

44. $f(x) = \dfrac{x}{(x - 1)^2}$

45. $f(x) = \sqrt{x^2 + 1} - x$

46. $f(x) = x \tan x, \quad -\pi/2 < x < \pi/2$

47. $f(x) = \ln(1 - \ln x)$

48. $f(x) = \dfrac{e^x}{1 + e^x}$

49. $f(x) = e^{-1/(x+1)}$

50. $f(x) = \ln(\tan^2 x)$

51–52 □

(a) Use a graph of f to estimate the maximum and minimum values. Then find the exact values.
(b) Estimate the value of x at which f increases most rapidly. Then find the exact value.

51. $f(x) = \dfrac{x + 1}{\sqrt{x^2 + 1}}$

52. $f(x) = x^2 e^{-x}$

53–54 □

(a) Use a graph of f to give a rough estimate of the intervals o concavity and the coordinates of the points of inflection.
(b) Use a graph of f'' to give better estimates.

53. $f(x) = 3x^5 - 40x^3 + 30x^2$

54. $f(x) = 2\cos x + \sin 2x, \quad 0 \leqslant x \leqslant 2\pi$

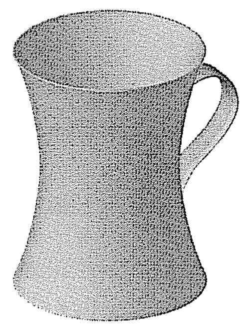
55–56 □ Estimate the intervals of concavity to one decimal pla by using a computer algebra system to compute and graph f''.

55. $f(x) = \dfrac{x^3 - 10x + 5}{\sqrt{x^2 + 4}}$

56. $f(x) = \dfrac{(x + 1)^3(x^2 + 5}{(x^3 + 1)(x^2 + 4}$

57. Let $K(t)$ be a measure of the knowledge you gain by studyi for a test for t hours. Which do you think is larger, $K(8) - K(7)$ or $K(3) - K(2)$? Is the graph of K concave upward or concave downward? Why?

58. Coffee is being poured into the mug shown in the figure at constant rate (measured in volume per unit time). Sketch a rough graph of the depth of the coffee in the mug as a func of time. Account for the shape of the graph in terms of con ity. What is the significance of the inflection point?



59. For the period from 1980 to 1994, the percentage of house- holds in the United States with at least one VCR has been modeled by the function

$$V(t) = \frac{75}{1 + 74e^{-0.6t}}$$

where the time t is measured in years since midyear 1980, $0 \leqslant t \leqslant 14$. Use a graph to estimate the time at which the number of VCRs was increasing most rapidly. Then use de tives to give a more accurate estimate.

60. The family of bell-shaped curves

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

occurs in probability and statistics, where it is called the *nc mal density function.* The constant μ is called the *mean* anc positive constant σ is called the *standard deviation.* For simplicity, let's scale the function so as to remove the facto $1/(\sigma\sqrt{2\pi})$ and let's analyze the special case where $\mu = 0$. we study the function

$$f(x) = e^{-x^2/(2\sigma^2)}$$

$$f(x) - x = -\frac{x}{x^2+1} = -\frac{\frac{1}{x}}{1+\frac{1}{x^2}} \to 0 \quad \text{as} \quad x \to \pm\infty$$

So the line $y = x$ is a slant asymptote.

E. $$f'(x) = \frac{3x^2(x^2+1) - x^3 \cdot 2x}{(x^2+1)^2} = \frac{x^2(x^2+3)}{(x^2+1)^2}$$

Since $f'(x) > 0$ for all x (except 0), f is increasing on $(-\infty, \infty)$.

F. Although $f'(0) = 0$, f' does not change sign at 0, so there is no local maximum or minimum.

G. $$f''(x) = \frac{(4x^3+6x)(x^2+1)^2 - (x^4+3x^2) \cdot 2(x^2+1)2x}{(x^2+1)^4} = \frac{2x(3-x^2)}{(x^2+1)^3}$$

Since $f''(x) = 0$ when $x = 0$ or $x = \pm\sqrt{3}$, we set up the following chart:

Interval	x	$3 - x^2$	$(x^2+1)^3$	$f''(x)$	f
$x < -\sqrt{3}$	$-$	$-$	$+$	$+$	CU on $(-\infty, -\sqrt{3})$
$-\sqrt{3} < x < 0$	$-$	$+$	$+$	$-$	CD on $(-\sqrt{3}, 0)$
$0 < x < \sqrt{3}$	$+$	$+$	$+$	$+$	CU on $(0, \sqrt{3})$
$x > \sqrt{3}$	$+$	$-$	$+$	$-$	CD on $(\sqrt{3}, \infty)$

The points of inflection are $(-\sqrt{3}, -3\sqrt{3}/4)$, $(0, 0)$, and $(\sqrt{3}, 3\sqrt{3}/4)$.

H. The graph of f is sketched in Figure 17. □

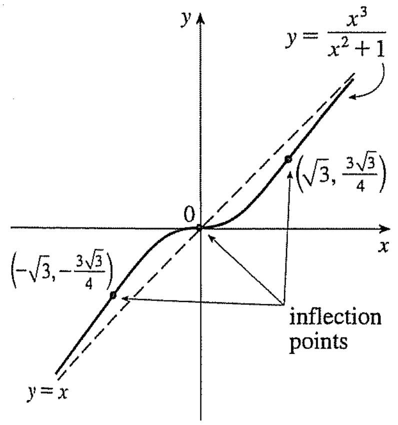


FIGURE 17

4.5 Exercises

1–50 □ Use the guidelines of this section to sketch the curve.

1. $y = x^3 + x$

2. $y = x^3 + 6x^2 + 9x$

3. $y = 2 - 15x + 9x^2 - x^3$

4. $y = 8x^2 - x^4$

5. $y = x^4 + 4x^3$

6. $y = 2 - x - x^9$

7. $y = \dfrac{x}{x-1}$

8. $y = \dfrac{x}{(x-1)^2}$

9. $y = \dfrac{1}{x^2-9}$

10. $y = \dfrac{x}{x^2-9}$

11. $y = \dfrac{x}{x^2+9}$

12. $y = \dfrac{x^2}{x^2+9}$

13. $y = \dfrac{1}{(x-1)(x+2)}$

14. $y = \dfrac{1}{x^2(x+3)}$

15. $y = \dfrac{1+x^2}{1-x^2}$

16. $y = \dfrac{x^3-1}{x^3+1}$

17. $y = \dfrac{1}{x^3-x}$

18. $y = \dfrac{1-x^2}{x^3}$

19. $y = x\sqrt{5-x}$

20. $y = \sqrt{x} - \sqrt{x-1}$

21. $y = \sqrt{x^2+1} - x$

22. $y = \sqrt{\dfrac{x}{x-5}}$

23. $y = \sqrt[4]{x^2-25}$

24. $y = x\sqrt{x^2-9}$

25. $y = \dfrac{\sqrt{1-x^2}}{x}$

26. $y = \dfrac{x+1}{\sqrt{x^2+1}}$

27. $y = x + 3x^{2/3}$

28. $y = x^{5/3} - 5x^{2/3}$

29. $y = x + \sqrt{|x|}$

30. $y = \sqrt[3]{(x^2-1)^2}$

31. $y = \cos x - \sin x$

32. $y = \sin x - \tan x$

33. $y = x \tan x, \quad -\pi/2 < x < \pi/2$

34. $y = 2x + \cot x, \quad 0 < x < \pi$

35. $y = \frac{1}{2}x - \sin x, \quad 0 < x < 3\pi$

36. $y = \cos^2 x - 2\sin x$

37. $y = 2\cos x + \sin^2 x$

38. $y = \sin x - x$

39. $y = \sin 2x - 2\sin x$

40. $y = \dfrac{\cos x}{2 + \sin x}$

41. $y = 1/(1 + e^{-x})$

42. $y = \ln(\cos x)$

43. $y = x \ln x$

44. $y = e^x/x$

45. $y = xe^{-x}$

46. $y = (\ln x)/x$

47. $y = \ln(x^2 - x)$

48. $y = x(\ln x)^2$

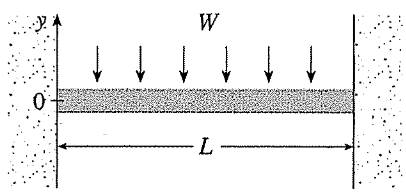
49. $y = xe^{-x^2}$

50. $y = e^x - 3e^{-x} - 4x$

51. The figure shows a beam of length L embedded in concrete walls. If a constant load W is distributed evenly along its length, the beam takes the shape of the deflection curve

$$y = -\frac{W}{24EI}x^4 + \frac{WL}{12EI}x^3 - \frac{WL^2}{24EI}x^2$$

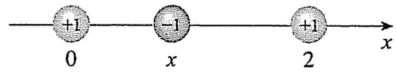
where E and I are positive constants. (E is Young's modulus of elasticity and I is the moment of inertia of a cross-section of the beam.) Sketch the graph of the deflection curve.



52. Coulomb's Law states that the force of attraction between two charged particles is directly proportional to the product of the charges and inversely proportional to the square of the distance between them. The figure shows particles with charge 1 located at positions 0 and 2 on a coordinate line and a particle with charge -1 at a position x between them. It follows from Coulomb's Law that the net force acting on the middle particle is

$$F(x) = -\frac{k}{x^2} - \frac{k}{(x-2)^2}$$

where k is a positive constant. Sketch the graph of the net fo function. What does the graph say about the force?



53–58 □ Use the guidelines of this section to sketch the curve. Ir guideline D find an equation of the slant asymptote.

53. $y = \dfrac{x^3}{x^2 - 1}$

54. $y = x - \dfrac{1}{x}$

55. $xy = x^2 + 4$

56. $y = e^x - x$

57. $y = \dfrac{1}{x - 1} - x$

58. $y = \dfrac{x^2}{2x + 5}$

59. Show that the lines $y = (b/a)x$ and $y = -(b/a)x$ are slant asymptotes of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$.

60. Let $f(x) = (x^3 + 1)/x$. Show that

$$\lim_{x\to\pm\infty} [f(x) - x^2] = 0$$

This shows that the graph of f approaches the graph of $y = $; and we say that the curve $y = f(x)$ is *asymptotic* to the parabola $y = x^2$. Use this fact to help sketch the graph of f.

61. Discuss the asymptotic behavior of $f(x) = (x^4 + 1)/x$ in the same manner as in Exercise 60. Then use your results to help sketch the graph of f.

62. Use the asymptotic behavior of $f(x) = \cos x + 1/x^2$ to sketc its graph without going through the curve-sketching procedur of this section.

4.6 Graphing with Calculus *and* Calculators

□ If you have not already read Section 1.4, you should do so now. In particular, it explains how to avoid some of the pitfalls of graphing devices by choosing appropriate viewing rectangles.

The method we used to sketch curves in the preceding section was a culmination of muc of our study of differential calculus. The graph was the final object that we produced. this section our point of view is completely different. Here we *start* with a graph produce by a graphing calculator or computer and then we refine it. We use calculus to make su that we reveal all the important aspects of the curve. And with the use of graphing devic we can tackle curves that would be far too complicated to consider without technolog The theme is the *interaction* between calculus and calculators.

EXAMPLE 1 □ Graph the polynomial $f(x) = 2x^6 + 3x^5 + 3x^3 - 2x^2$. Use the graphs o f' and f'' to estimate all maximum and minimum points and intervals of concavity.

SOLUTION If we specify a domain but not a range, many graphing devices will deduce a suitable range from the values computed. Figure 1 shows the plot from one such device if we specify that $-5 \leq x \leq 5$. Although this viewing rectangle is useful for showing

Then $$\ln y = \ln[(1 + \sin 4x)^{\cot x}] = \cot x \ln(1 + \sin 4x)$$

so l'Hospital's Rule gives

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(1 + \sin 4x)}{\tan x} = \lim_{x\to 0^+} \frac{\dfrac{4\cos 4x}{1 + \sin 4x}}{\sec^2 x} = 4$$

So far we have computed the limit of $\ln y$, but what we want is the limit of y. To find this we use the fact that $y = e^{\ln y}$:

$$\lim_{x\to 0^+} (1 + \sin 4x)^{\cot x} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^4$$

□ The graph of the function $y = x^x$, $x > 0$, is shown in Figure 6. Notice that although 0^0 is not defined, the values of the function approach 1 as $x \to 0^+$. This confirms the result of Example 9.

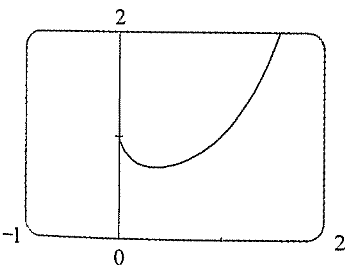


FIGURE 6

EXAMPLE 9 □ Find $\lim_{x\to 0^+} x^x$.

SOLUTION Notice that this limit is indeterminate since $0^x = 0$ for any $x > 0$ but $x^0 = 1$ for any $x \neq 0$. We could proceed as in Example 8 or by writing the function as an exponential:

$$x^x = (e^{\ln x})^x = e^{x \ln x}$$

In Example 6 we used l'Hospital's Rule to show that

$$\lim_{x\to 0^+} x \ln x = 0$$

Therefore

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = e^0 = 1$$

4.4 Exercises

1–4 □ Given that

$$\lim_{x\to a} f(x) = 0 \qquad \lim_{x\to a} g(x) = 0 \qquad \lim_{x\to a} h(x) = 1$$

$$\lim_{x\to a} p(x) = \infty \qquad \lim_{x\to a} q(x) = \infty$$

which of the following limits are indeterminate forms? For those that are not an indeterminate form, evaluate the limit where possible.

1. (a) $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ (b) $\lim_{x\to a} \dfrac{f(x)}{p(x)}$ (c) $\lim_{x\to a} \dfrac{h(x)}{p(x)}$

(d) $\lim_{x\to a} \dfrac{p(x)}{f(x)}$ (e) $\lim_{x\to a} \dfrac{p(x)}{q(x)}$

2. (a) $\lim_{x\to a} [f(x)p(x)]$ (b) $\lim_{x\to a} [h(x)p(x)]$

(c) $\lim_{x\to a} [p(x)q(x)]$

3. (a) $\lim_{x\to a} [f(x) - p(x)]$ (b) $\lim_{x\to a} [p(x) - q(x)]$

(c) $\lim_{x\to a} [p(x) + q(x)]$

4. (a) $\lim_{x\to a} [f(x)]^{g(x)}$ (b) $\lim_{x\to a} [f(x)]^{p(x)}$ (c) $\lim_{x\to a} [h(x)]^{p(x)}$

(d) $\lim_{x\to a} [p(x)]^{f(x)}$ (e) $\lim_{x\to a} [p(x)]^{q(x)}$ (f) $\lim_{x\to a} \sqrt[q(x)]{p(x)}$

5–66 □ Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, use it. If l'Hospital's Rule doesn't apply, explain why.

5. $\lim_{x\to -1} \dfrac{x^2 - 1}{x + 1}$

6. $\lim_{x\to -2} \dfrac{x + 2}{x^2 + 3x + 2}$

7. $\lim_{x\to 1} \dfrac{x^9 - 1}{x^5 - 1}$

8. $\lim_{x\to 1} \dfrac{x^a - 1}{x^b - 1}$

9. $\lim_{x \to 0} \frac{e^x - 1}{\sin x}$

10. $\lim_{x \to 0} \frac{x + \tan x}{\sin x}$

11. $\lim_{x \to 0} \frac{\sin x}{x^3}$

12. $\lim_{x \to \pi} \frac{\tan x}{x}$

13. $\lim_{x \to 0} \frac{\tan px}{\tan qx}$

14. $\lim_{x \to 3\pi/2} \frac{\cos x}{x - (3\pi/2)}$

15. $\lim_{x \to \infty} \frac{\ln x}{x}$

16. $\lim_{x \to \infty} \frac{e^x}{x}$

17. $\lim_{x \to 0^+} \frac{\ln x}{x}$

18. $\lim_{x \to \infty} \frac{\ln \ln x}{x}$

19. $\lim_{t \to 0} \frac{5^t - 3^t}{t}$

20. $\lim_{t \to 16} \frac{\sqrt[4]{t} - 2}{t - 16}$

21. $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$

22. $\lim_{x \to 0} \frac{e^x - 1 - x - (x^2/2)}{x^3}$

23. $\lim_{x \to \infty} \frac{e^x}{x^3}$

24. $\lim_{x \to \infty} \frac{(\ln x)^3}{x^2}$

25. $\lim_{x \to 0} \frac{\sin^{-1}x}{x}$

26. $\lim_{x \to 0} \frac{\sin x}{\sinh x}$

27. $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

28. $\lim_{x \to 0} \frac{\sin x - x}{x^3}$

29. $\lim_{x \to 0} \frac{\sin x}{e^x}$

30. $\lim_{x \to 0} \frac{\cos mx - \cos nx}{x^2}$

31. $\lim_{x \to 0} \frac{\tan \alpha x}{x}$

32. $\lim_{x \to 0} \frac{x}{\tan^{-1}(4x)}$

33. $\lim_{x \to \infty} \frac{x}{\ln(1 + 2e^x)}$

34. $\lim_{x \to 0} \frac{x + \tan 2x}{x - \tan 2x}$

35. $\lim_{x \to 0} \frac{\tan 2x}{\tanh 3x}$

36. $\lim_{x \to 0} \frac{1 - e^{-2x}}{\sec x}$

37. $\lim_{x \to 0} \frac{2x - \sin^{-1}x}{2x + \cos^{-1}x}$

38. $\lim_{x \to 0} \frac{2x - \sin^{-1}x}{2x + \tan^{-1}x}$

39. $\lim_{x \to 0^+} \sqrt{x} \ln x$

40. $\lim_{x \to -\infty} x^2 e^x$

41. $\lim_{x \to \infty} e^{-x} \ln x$

42. $\lim_{x \to (\pi/2)^-} \sec 7x \cos 3x$

43. $\lim_{x \to \infty} x^3 e^{-x^2}$

44. $\lim_{x \to 0^+} \sqrt{x} \sec x$

45. $\lim_{x \to \pi} (x - \pi) \cot x$

46. $\lim_{x \to 1^+} (x - 1) \tan(\pi x/2)$

47. $\lim_{x \to 0} \left(\frac{1}{x^4} - \frac{1}{x^2}\right)$

48. $\lim_{x \to 0} (\csc x - \cot x)$

49. $\lim_{x \to 0} \left(\frac{1}{x} - \csc x\right)$

50. $\lim_{x \to 1} \left(\frac{1}{\ln x} - \frac{1}{x - 1}\right)$

51. $\lim_{x \to \infty} \left(x - \sqrt{x^2 - 1}\right)$

52. $\lim_{x \to \infty} \left(\sqrt{x^2 + x + 1} - \sqrt{x^2 - x}\right)$

53. $\lim_{x \to \infty} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$

54. $\lim_{x \to \infty} (xe^{1/x} - x)$

55. $\lim_{x \to 0^+} x^{\sin x}$

56. $\lim_{x \to 0^+} (\sin x)^{\tan x}$

57. $\lim_{x \to 0} (1 - 2x)^{1/x}$

58. $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^{bx}$

59. $\lim_{x \to \infty} \left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^x$

60. $\lim_{x \to \infty} x^{(\ln 2)/(1 + \ln x)}$

61. $\lim_{x \to \infty} x^{1/x}$

62. $\lim_{x \to \infty} (e^x + x)^{1/x}$

63. $\lim_{x \to \infty} \left(\frac{x}{x + 1}\right)^x$

64. $\lim_{x \to 0} (\cos 3x)^{5/x}$

65. $\lim_{x \to 0^+} (-\ln x)^x$

66. $\lim_{x \to \infty} \left(\frac{2x - 3}{2x + 5}\right)^{2x+1}$

67–68 □ Use a graph to estimate the value of the limit. Then use l'Hospital's Rule to find the exact value.

67. $\lim_{x \to \infty} x[\ln(x + 5) - \ln x]$

68. $\lim_{x \to \pi/4} (\tan x)^{\tan 2x}$

69–70 □ Illustrate l'Hospital's Rule by graphing both $f(x)/g(x)$ and $f'(x)/g'(x)$ near $x = 0$ to see that these ratios have the same limit as $x \to 0$. Also calculate the exact value of the limit.

69. $f(x) = e^x - 1, \quad g(x) = x^3 + 4x$

70. $f(x) = 2x \sin x, \quad g(x) = \sec x - 1$

71. Prove that

$$\lim_{x \to \infty} \frac{e^x}{x^n} = \infty$$

for any positive integer n. This shows that the exponential function approaches infinity faster than any power of x.

72. Prove that

$$\lim_{x \to \infty} \frac{\ln x}{x^p} = 0$$

for any number $p > 0$. This shows that the logarithmic function approaches ∞ more slowly than any power of x.

73. If an initial amount A_0 of money is invested at an interest rate i compounded n times a year, the value of the investment after t years is

$$A = A_0\left(1 + \frac{i}{n}\right)^{nt}$$

If we let $n \to \infty$, we refer to the *continuous compounding* of interest. Use l'Hospital's Rule to show that if interest is compounded continuously, then the amount after n years is

$$A = A_0 e^{it}$$

74. If an object with mass m is dropped from rest, one model for its speed v after t seconds, taking air resistance into account, is

$$v = \frac{mg}{c}(1 - e^{-ct/m})$$

where g is the acceleration due to gravity and c is a positive constant. (In Chapter 9 we will be able to deduce this equation from the assumption that the air resistance is proportional to the speed of the object.)

(a) Calculate $\lim_{t \to \infty} v$. What is the meaning of this limit?

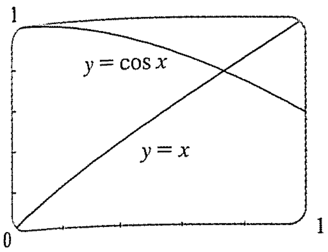


FIGURE 6

imation. Then Newton's method gives

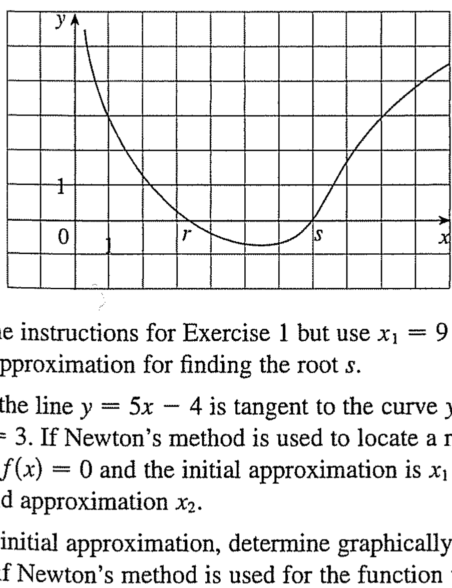
$$x_2 \approx 0.73911114$$
$$x_3 \approx 0.73908513$$
$$x_4 \approx 0.73908513$$

and so we obtain the same answer as before, but with one fewer step.

You might wonder why we bother at all with Newton's method if a graphing device is available. Isn't it easier to zoom in repeatedly and find the roots as we did in Section 1.4? If only one or two decimal places of accuracy are required, then indeed Newton's method is inappropriate and a graphing device suffices. But if six or eight decimal places are required, then repeated zooming becomes tiresome. It is usually faster and more efficient to use a computer and Newton's method in tandem—the graphing device to get started and Newton's method to finish.

4.9 Exercises

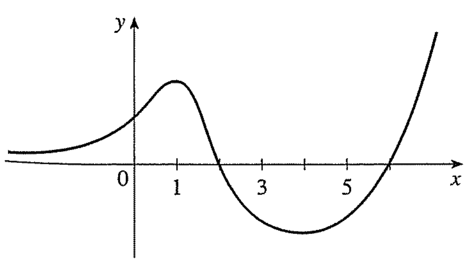
1. The figure shows the graph of a function f. Suppose that Newton's method is used to approximate the root r of the equation $f(x) = 0$ with initial approximation $x_1 = 1$. Draw the tangent lines that are used to find x_2 and x_3, and estimate the numerical values of x_2 and x_3.



2. Follow the instructions for Exercise 1 but use $x_1 = 9$ as the starting approximation for finding the root s.

3. Suppose the line $y = 5x - 4$ is tangent to the curve $y = f(x)$ when $x = 3$. If Newton's method is used to locate a root of the equation $f(x) = 0$ and the initial approximation is $x_1 = 3$, find the second approximation x_2.

4. For each initial approximation, determine graphically what happens if Newton's method is used for the function whose graph is shown.
(a) $x_1 = 0$ (b) $x_1 = 1$ (c) $x_1 = 3$
(d) $x_1 = 4$ (e) $x_1 = 5$



5–8 □ Use Newton's method with the specified initial approximation x_1 to find x_3, the third approximation to the root of the given equation. (Give your answer to four decimal places.)

5. $x^3 + x + 1 = 0, \quad x_1 = -1$

6. $x^3 - x^2 - 1 = 0, \quad x_1 = 1$

7. $x^4 - 20 = 0, \quad x_1 = 2$

8. $x^7 - 100 = 0, \quad x_1 = 2$

9–10 □ Use Newton's method to approximate the given number correct to eight decimal places.

9. $\sqrt[3]{30}$

10. $\sqrt[7]{1000}$

11–14 □ Use Newton's method to approximate the indicated root of the equation correct to six decimal places.

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11. The root of $2x^3 - 6x^2 + 3x + 1 = 0$ in the interval $[2, 3]$

12. The root of $x^4 + x - 4 = 0$ in the interval $[1, 2]$

13. The positive root of $2 \sin x = x$

14. The root of $\tan x = x$ in the interval $(\pi/2, 3\pi/2)$

15–20 □ Use Newton's method to find all roots of the equation correct to six decimal places.

15. $x^3 = 4x - 1$

16. $e^x = 3 - 2x$

17. $\tan^{-1}x = 1 - x$

18. $\sqrt{x + 3} = x^2$

19. $2 \cos x = 2 - x$

20. $\sin \pi x = x$

21–26 □ Use Newton's method to find all the roots of the equation correct to eight decimal places. Start by drawing a graph to find initial approximations.

21. $x^5 - x^4 - 5x^3 - x^2 + 4x + 3 = 0$