MST 10010: Calculus I

Exercise Set 4

Do only the following problems:

Section 3.5 (pages 221-222): 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 43, 45, 47 (a), 49 (a).

Section 3.6 (pages 230-231): 1, 3, 5, 7, 9, 11, 13, 15, 17, 25, 27, 29, 31 (a).

Section 3.7 (pages 237-238): 5, 7, 9, 11, 13, 15, 17, 19, 23, 25, 29, 31, 39.

Section 3.8 (pages 245-246): 3, 5, 7, 9, 11, 13, 15, 19, 21, 23, 25, 27, 29, 31, 35, 37, 39, 41, 43, 45, 47.

So if we denote by ε the difference between the difference quotient and the derivative, we obtain

$$\lim_{\Delta x \to 0} \varepsilon = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} - f'(a) \right) = f'(a) - f'(a) = 0$$

But $$\varepsilon = \frac{\Delta y}{\Delta x} - f'(a) \quad \Rightarrow \quad \Delta y = f'(a)\,\Delta x + \varepsilon\,\Delta x$$

Thus, for a differentiable function f, we can write

7 $$\Delta y = f'(a)\,\Delta x + \varepsilon\,\Delta x \qquad \text{where } \varepsilon \to 0 \text{ as } \Delta x \to 0$$

This property of differentiable functions is what enables us to prove the Chain Rule.

Proof of the Chain Rule Suppose $u = g(x)$ is differentiable at a and $y = f(u)$ is differentiable at $b = g(a)$. If Δx is an increment in x and Δu and Δy are the corresponding increments in u and y, then we can use Equation 7 to write

8 $$\Delta u = g'(a)\,\Delta x + \varepsilon_1\,\Delta x = [g'(a) + \varepsilon_1]\,\Delta x$$

where $\varepsilon_1 \to 0$ as $\Delta x \to 0$. Similarly

9 $$\Delta y = f'(b)\,\Delta u + \varepsilon_2\,\Delta u = [f'(b) + \varepsilon_2]\,\Delta u$$

where $\varepsilon_2 \to 0$ as $\Delta u \to 0$. If we now substitute the expression for Δu from Equation 8 into Equation 9, we get

$$\Delta y = [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]\,\Delta x$$

so $$\frac{\Delta y}{\Delta x} = [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]$$

As $\Delta x \to 0$, Equation 8 shows that $\Delta u \to 0$. So both $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $\Delta x \to 0$. Therefore

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]$$
$$= f'(b)g'(a) = f'(g(a))g'(a)$$

This proves the Chain Rule. □

3.5 Exercises

1–6 □ Write the composite function in the form $f(g(x))$. [Identify the inner function $u = g(x)$ and the outer function $y = f(u)$.] Then find the derivative dy/dx.

Resources / Module 4 / Trigonometric Models / Chain Rule Practice

1. $y = (x^2 + 4x + 6)^5$

2. $y = \tan 3x$

3. $y = \cos(\tan x)$

4. $y = \sqrt[3]{1 + x^3}$

5. $y = e^{\sqrt{x}}$

6. $y = \sin(e^x)$

7–42 □ Find the derivative of the function.

7. $F(x) = (x^3 + 4x)^7$

8. $F(x) = (x^2 - x + 1)^3$

9. $g(x) = \sqrt{x^2 - 7x}$

10. $f(t) = \dfrac{1}{(t^2 - 2t - 5)^4}$

11. $h(t) = \left(t - \dfrac{1}{t} \right)^{3/2}$

12. $f(t) = \sqrt[3]{1 + \tan t}$

13. $y = \cos(a^3 + x^3)$

14. $y = a^3 + \cos^3 x$

15. $y = e^{-mx}$

16. $y = 4 \sec 5x$

17. $G(x) = (3x - 2)^{10}(5x^2 - x + 1)^{12}$

18. $g(t) = (6t^2 + 5)^3(t^3 - 7)^4$

19. $y = (2x - 5)^4(8x^2 - 5)^{-3}$

20. $y = (x^2 + 1)\sqrt[3]{x^2 + 2}$

21. $y = xe^{-x^2}$

22. $y = e^{-5x} \cos 3x$

23. $F(y) = \left(\dfrac{y - 6}{y + 7}\right)^3$

24. $s(t) = \sqrt[4]{\dfrac{t^3 + 1}{t^3 - 1}}$

25. $f(z) = \dfrac{1}{\sqrt[5]{2z - 1}}$

26. $f(x) = \dfrac{x}{\sqrt{7 - 3x}}$

27. $y = \tan(\cos x)$

28. $y = \dfrac{\sin^2 x}{\cos x}$

29. $y = 5^{-1/x}$

30. $y = \sqrt{1 + 2 \tan x}$

31. $y = \sin^3 x + \cos^3 x$

32. $y = \sin^2(\cos kx)$

33. $y = (1 + \cos^2 x)^6$

34. $y = x \sin \dfrac{1}{x}$

35. $y = \dfrac{e^{3x}}{1 + e^x}$

36. $y = e^{5 \sin \theta}$

37. $y = e^{x \cos x}$

38. $y = \sin(\sin(\sin x))$

39. $y = \sqrt{x + \sqrt{x}}$

40. $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

41. $y = \sin(\tan \sqrt{\sin x})$

42. $y = 2^{3^{x^2}}$

43–46 □ Find an equation of the tangent line to the curve at the given point.

43. $y = \dfrac{8}{\sqrt{4 + 3x}}$, $(4, 2)$

44. $y = \sin x + \cos 2x$, $(\pi/6, 1)$

45. $y = \sin(\sin x)$, $(\pi, 0)$

46. $y = 10^x$, $(1, 10)$

47. (a) Find an equation of the tangent line to the curve $y = 2/(1 + e^{-x})$ at the point $(0, 1)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

48. (a) The curve $y = |x|/\sqrt{2 - x^2}$ is called a **bullet-nose curve.** Find an equation of the tangent line to this curve at the point $(1, 1)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

49. (a) If $f(x) = \sqrt{1 - x^2}/x$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

50. (a) If $f(x) = 1/(\cos^2 \pi x + 9 \sin^2 \pi x)$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

51. Find all points on the graph of the function $f(x) = 2 \sin x + \sin^2 x$ at which the tangent line is horizontal.

52. Find the x-coordinates of all points on the curve $y = \sin 2x - 2 \sin x$ at which the tangent line is horizontal.

53. Suppose that $F(x) = f(g(x))$ and $g(3) = 6$, $g'(3) = 4$, $f'(3) = 2$, and $f'(6) = 7$. Find $F'(3)$.

54. Suppose that $w = u \circ v$ and $u(0) = 1$, $v(0) = 2$, $u'(0) = 3$, $u'(2) = 4$, $v'(0) = 5$, and $v'(2) = 6$. Find $w'(0)$.

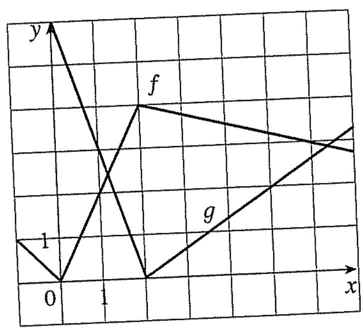
55. A table of values for f, g, f', and g' is given.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

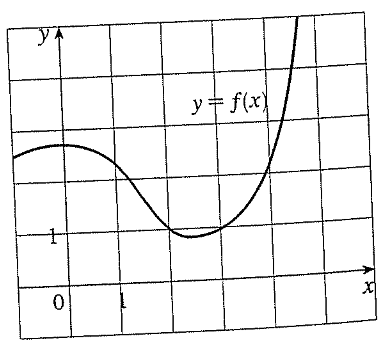
(a) If $h(x) = f(g(x))$, find $h'(1)$.
(b) If $H(x) = g(f(x))$, find $H'(1)$.

56. Let f and g be the functions in Exercise 55.
(a) If $F(x) = f(f(x))$, find $F'(2)$.
(b) If $G(x) = g(g(x))$, find $G'(3)$.

57. If f and g are the functions whose graphs are shown, let $u(x) = f(g(x))$, $v(x) = g(f(x))$, and $w(x) = g(g(x))$. Find ea derivative, if it exists. If it does not exist, explain why.
(a) $u'(1)$ (b) $v'(1)$ (c) $w'(1)$



58. If f is the function whose graph is shown, let $h(x) = f(f($ and $g(x) = f(x^2)$. Use the graph of f to estimate the value each derivative.
(a) $h'(2)$ (b) $g'(2)$



EXAMPLE 5 □ Differentiate (a) $y = \dfrac{1}{\sin^{-1}x}$ and (b) $f(x) = x\tan^{-1}\sqrt{x}$.

SOLUTION

(a)
$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x)^{-1} = -(\sin^{-1}x)^{-2}\frac{d}{dx}(\sin^{-1}x)$$
$$= -\frac{1}{(\sin^{-1}x)^2\sqrt{1-x^2}}$$

(b)
$$f'(x) = \tan^{-1}\sqrt{x} + x\frac{1}{1+(\sqrt{x})^2}\left(\tfrac{1}{2}x^{-1/2}\right)$$
$$= \tan^{-1}\sqrt{x} + \frac{\sqrt{x}}{2(1+x)}$$
□

The inverse trigonometric functions that occur most frequently are the ones that we have just discussed. The derivatives of the remaining four are given in the following table. The proofs of the formulas are left as exercises.

□ The formulas for the derivatives of $\csc^{-1}x$ and $\sec^{-1}x$ depend on the definitions that are used for these functions. See Exercise 54.

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

3.6 Exercises

1–4 □
(a) Find y' by implicit differentiation.
(b) Solve the equation explicitly for y and differentiate to get y' in terms of x.
(c) Check that your solutions to parts (a) and (b) are consistent by substituting the expression for y into your solution for part (a).

1. $xy + 2x + 3x^2 = 4$
2. $4x^2 + 9y^2 = 36$
3. $\dfrac{1}{x} + \dfrac{1}{y} = 1$
4. $\sqrt{x} + \sqrt{y} = 4$

5–20 □ Find dy/dx by implicit differentiation.

5. $x^2 + y^2 = 1$
6. $x^2 - y^2 = 1$
7. $x^3 + x^2y + 4y^2 = 6$
8. $x^2 - 2xy + y^3 = c$
9. $x^2y + xy^2 = 3x$
10. $y^5 + x^2y^3 = 1 + ye^{x^2}$
11. $\dfrac{y}{x-y} = x^2 + 1$
12. $\sqrt{x+y} + \sqrt{xy} = 6$
13. $\sqrt{xy} = 1 + x^2y$
14. $\sqrt{1 + x^2y^2} = 2xy$
15. $4\cos x\sin y = 1$
16. $x\sin y + \cos 2y = \cos y$
17. $\cos(x - y) = xe^x$
18. $x\cos y + y\cos x = 1$
19. $xy = \cot(xy)$
20. $\sin x + \cos y = \sin x\cos y$

21. If $x[f(x)]^3 + xf(x) = 6$ and $f(3) = 1$, find $f'(3)$.

22. If $[g(x)]^2 + 12x = x^2g(x)$ and $g(4) = 12$, find $g'(4)$.

23–24 □ Regard y as the independent variable and x as the dependent variable and use implicit differentiation to find dx/dy.

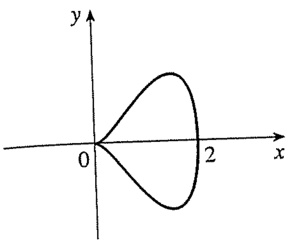
23. $y^4 + x^2y^2 + yx^4 = y + 1$
24. $(x^2 + y^2)^2 = ax^2y$

25–30 □ Find an equation of the tangent line to the curve at the given point.

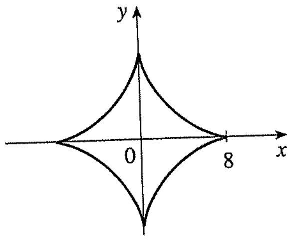
25. $\frac{x^2}{16} - \frac{y^2}{9} = 1$, $\left(-5, \frac{9}{4}\right)$ (hyperbola)

26. $\frac{x^2}{9} + \frac{y^2}{36} = 1$, $\left(-1, 4\sqrt{2}\right)$ (ellipse)

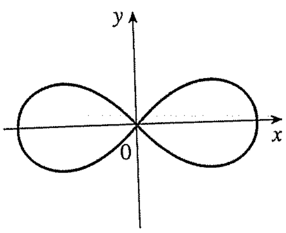
27. $y^2 = x^3(2 - x)$
$(1, 1)$
(piriform)



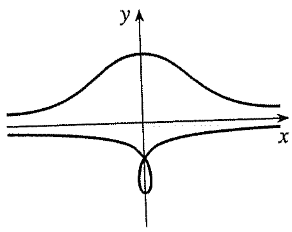
28. $x^{2/3} + y^{2/3} = 4$
$\left(-3\sqrt{3}, 1\right)$
(astroid)



29. $2(x^2 + y^2)^2 = 25(x^2 - y^2)$
$(3, 1)$
(lemniscate)



30. $x^2y^2 = (y + 1)^2(4 - y^2)$
$(0, -2)$
(conchoid of Nicomedes)



31. (a) The curve with equation $y^2 = 5x^4 - x^2$ is called a **kampyle of Eudoxus**. Find an equation of the tangent line to this curve at the point $(1, 2)$.

(b) Illustrate part (a) by graphing the curve and the tangent line on a common screen. (If your graphing device will graph implicitly defined curves, then use that capability. If not, you can still graph this curve by graphing its upper and lower halves separately.)

32. (a) The curve with equation $y^2 = x^3 + 3x^2$ is called the **Tschirnhausen cubic**. Find an equation of the tangent line to this curve at the point $(1, -2)$.

(b) At what points does this curve have a horizontal tangent?

(c) Illustrate parts (a) and (b) by graphing the curve and the tangent lines on a common screen.

33. Fanciful shapes can be created by using the implicit plotting capabilities of computer algebra systems.

(a) Graph the curve with equation

$$y(y^2 - 1)(y - 2) = x(x - 1)(x - 2)$$

At how many points does this curve have horizontal tangents? Estimate the x-coordinates of these points.

(b) Find equations of the tangent lines at the points $(0, 1)$ and $(0, 2)$.

(c) Find the exact x-coordinates of the points in part (a).

(d) Create even more fanciful curves by modifying the equation in part (a).

34. (a) The curve with equation

$$2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2$$

has been likened to a bouncing wagon. Use a computer algebra system to graph this curve and discover why.

(b) At how many points does this curve have horizontal tangent lines? Find the x-coordinates of these points.

35. Find the points on the lemniscate in Exercise 29 where the tangent is horizontal.

36. Show by implicit differentiation that the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) is

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1$$

37. Find an equation of the tangent line to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0).

38. Show that the sum of the x- and y-intercepts of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is equal to c.

39. Show, using implicit differentiation, that any tangent line at a point P to a circle with center O is perpendicular to the radius OP.

40. The Power Rule can be proved using implicit differentiation for the case where n is a rational number, $n = p/q$, and $y = f(x) = x^n$ is assumed beforehand to be a differentiable function. If $y = x^{p/q}$, then $y^q = x^p$. Use implicit differentiatio to show that

$$y' = \frac{p}{q}x^{(p/q)-1}$$

41–50 □ Find the derivative of the function. Simplify where possible.

41. $y = \sin^{-1}(x^2)$

42. $y = (\sin^{-1}x)^2$

43. $y = \tan^{-1}(e^x)$

44. $h(x) = \sqrt{1 - x^2}$ arcsin

45. $H(x) = (1 + x^2)\arctan x$

46. $y = \tan^{-1}(x - \sqrt{1 + x}$

47. $g(t) = \sin^{-1}(4/t)$

48. $y = x\cos^{-1}x - \sqrt{1 - .}$

49. $y = x^2\cot^{-1}(3x)$

50. $y = \arctan(\cos\theta)$

51–52 □ Find $f'(x)$. Check that your answer is reasonable by co paring the graphs of f and f'.

51. $f(x) = e^x - x^2\arctan x$

52. $f(x) = x\arcsin(1 - x$

$$D^4 \cos x = \cos x$$

$$D^5 \cos x = -\sin x$$

We see that the successive derivatives occur in a cycle of length 4 and, in particular, $D^n \cos x = \cos x$ whenever n is a multiple of 4. Therefore

$$D^{24} \cos x = \cos x$$

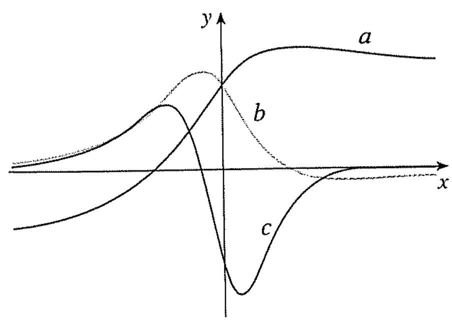
and, differentiating three more times, we have

$$D^{27} \cos x = \sin x$$

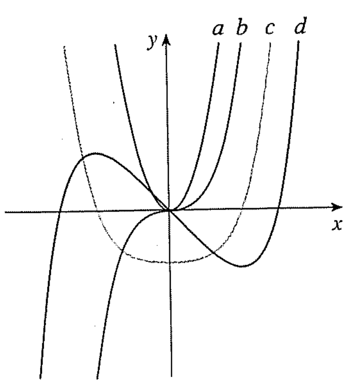
We have seen that one application of second and third derivatives occurs in analyzi the motion of objects using acceleration and jerk. We will investigate another applicati of second derivatives in Exercise 62 and in Section 4.3, where we show how knowledge f'' gives us information about the shape of the graph of f. In Chapter 11 we will see h second and higher derivatives enable us to represent functions as sums of infinite serie

3.7 Exercises

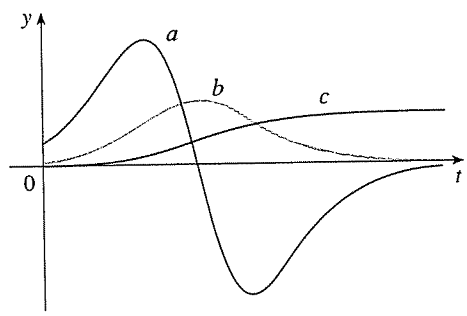
1. The figure shows the graphs of f, f', and f''. Identify each curve, and explain your choices.



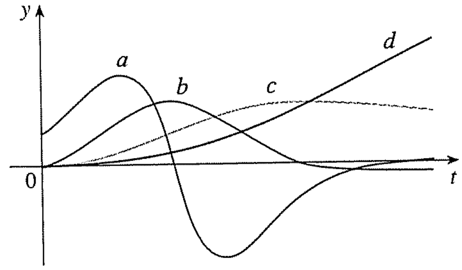
2. The figure shows graphs of f, f', f'', and f'''. Identify each curve, and explain your choices.



3. The figure shows the graphs of three functions. One is the position function of a car, one is the velocity of the car, and one is its acceleration. Identify each curve, and explain your choices.



4. The figure shows the graphs of four functions. One is the position function of a car, one is the velocity of the car, one its acceleration, and one is its jerk. Identify each curve, and explain your choices.



5–20 □ Find the first and second derivatives of the function.

5. $f(x) = x^5 + 6x^2 - 7x$

6. $f(t) = t^8 - 7t^6 + 2t^4$

7. $y = \cos 2\theta$

8. $y = \theta \sin \theta$

9. $h(x) = \sqrt{x^2 + 1}$

10. $G(r) = \sqrt{r} + \sqrt[3]{r}$

11. $F(s) = (3s + 5)^8$

12. $g(u) = \dfrac{1}{\sqrt{1 - u}}$

13. $y = \dfrac{x}{1 - x}$

14. $y = xe^{cx}$

15. $y = (1 - x^2)^{3/4}$

16. $y = \dfrac{x^2}{x + 1}$

17. $H(t) = \tan 3t$

18. $g(s) = s^2 \cos s$

19. $g(t) = t^3 e^{5t}$

20. $h(x) = \tan^{-1}(x^2)$

21. (a) If $f(x) = 2\cos x + \sin^2 x$, find $f'(x)$ and $f''(x)$.
(b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f, f', and f''.

22. (a) If $f(x) = e^x - x^3$, find $f'(x)$ and $f''(x)$.
(b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f, f', and f''.

23–24 □ Find y'''.

23. $y = \sqrt{2x + 3}$

24. $y = \dfrac{1 - x}{1 + x}$

25. If $f(x) = (2 - 3x)^{-1/2}$, find $f(0)$, $f'(0)$, $f''(0)$, and $f'''(0)$.

26. If $g(t) = (2 - t^2)^6$, find $g(0)$, $g'(0)$, $g''(0)$, and $g'''(0)$.

27. If $f(\theta) = \cot\theta$, find $f'''(\pi/6)$.

28. If $g(x) = \sec x$, find $g'''(\pi/4)$.

29–32 □ Find y'' by implicit differentiation.

29. $x^3 + y^3 = 1$

30. $\sqrt{x} + \sqrt{y} = 1$

31. $x^2 + xy + y^2 = 1$

32. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

33–37 □ Find a formula for $f^{(n)}(x)$.

33. $f(x) = x^n$

34. $f(x) = \dfrac{1}{(1 - x)^2}$

35. $f(x) = e^{2x}$

36. $f(x) = \sqrt{x}$

37. $f(x) = \dfrac{1}{3x^3}$

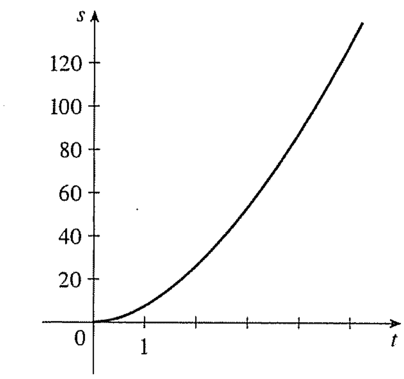
38–40 □ Find the given derivative by finding the first few derivatives and observing the pattern that occurs.

38. $D^{99} \sin x$

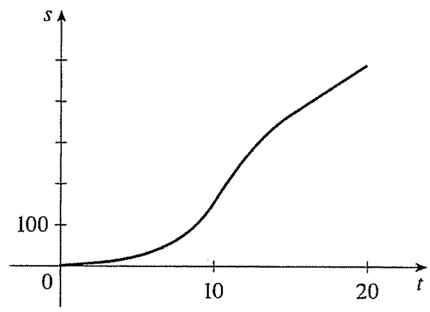
39. $D^{50} \cos 2x$

40. $D^{1000} xe^{-x}$

41. A car starts from rest and the graph of its position function is shown in the figure, where s is measured in feet and t in seconds. Use it to graph the velocity and estimate the acceleration at $t = 2$ seconds from the velocity graph. Then sketch a graph of the acceleration function.



42. (a) The graph of a position function of a car is shown, where s is measured in feet and t in seconds. Use it to graph the velocity and acceleration of the car. What is the acceleration at $t = 10$ seconds?



(b) Use the acceleration curve from part (a) to estimate the jerk at $t = 10$ seconds. What are the units for jerk?

43–46 □ The equation of motion is given for a particle, where s is in meters and t is in seconds. Find (a) the velocity and acceleration as functions of t, (b) the acceleration after 1 second, and (c) the acceleration at the instants when the velocity is 0.

43. $s = t^3 - 3t$

44. $s = t^2 - t + 1$

45. $s = \sin 2\pi t$

46. $s = 2t^3 - 7t^2 + 4t + 1$

47–48 □ An equation of motion is given, where s is in meters and t in seconds. Find (a) the times at which the acceleration is 0 and (b) the displacement and velocity at these times.

47. $s = t^4 - 4t^3 + 2$

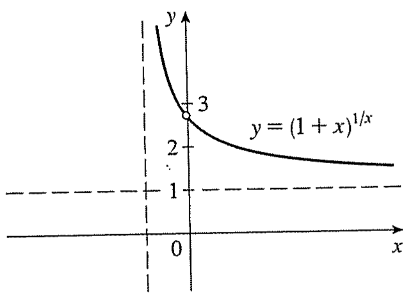
48. $s = 2t^3 - 9t^2$

Therefore

5 $$\lim_{x \to 0} (1 + x)^{1/x} = e$$

Formula 5 is illustrated by the graph of the function $y = (1 + x)^{1/x}$ in Figure 4 and a table of values for small values of x. This illustrates the fact that, correct to seven decimal places,

$$e \approx 2.7182818$$



x	$(1 + x)^{1/x}$
0.1	2.59374246
0.01	2.70481383
0.001	2.71692393
0.0001	2.71814593
0.00001	2.71826824
0.000001	2.71828047
0.0000001	2.71828169
0.00000001	2.71828181

FIGURE 4

If we put $n = 1/x$ in Formula 5, then $n \to \infty$ as $x \to 0^+$ and so an alternative expression for e is

6 $$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

3.8 Exercises

1. Explain why the natural logarithmic function $y = \ln x$ is used much more frequently in calculus than the other logarithmic functions $y = \log_a x$.

2–20 □ Differentiate the function.

2. $f(x) = \ln(2 - x)$
3. $f(\theta) = \ln(\cos\theta)$
4. $f(x) = \cos(\ln x)$
5. $f(x) = \log_3(x^2 - 4)$
6. $f(x) = \log_{10}\left(\dfrac{x}{x - 1}\right)$
7. $F(x) = \ln\sqrt{x}$
8. $G(x) = \sqrt[3]{\ln x}$
9. $f(x) = \sqrt{x}\ln x$
10. $f(t) = \dfrac{1 + \ln t}{1 - \ln t}$
11. $g(x) = \ln\dfrac{a - x}{a + x}$
12. $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$
13. $F(x) = e^x \ln x$
14. $h(y) = \ln(y^3 \sin y)$
15. $y = \dfrac{\ln x}{1 + x}$
16. $y = (\ln \tan x)^2$
17. $y = \ln|x^3 - x^2|$
18. $G(u) = \ln\sqrt{\dfrac{3u + 2}{3u - 2}}$
19. $y = \ln(e^{-x} + xe^{-x})$
20. $y = \ln(x + \ln x)$

21–24 □ Find y' and y''.

21. $y = x\ln x$
22. $y = \ln(1 + x^2)$
23. $y = \log_{10} x$
24. $y = \ln(\sec x + \tan x)$

25–28 □ Differentiate f and find the domain of f.

25. $f(x) = \ln(2x + 1)$

26. $f(x) = \dfrac{1}{1 + \ln x}$

27. $f(x) = x^2 \ln(1 - x^2)$

28. $f(x) = \ln \ln \ln x$

29. If $f(x) = \dfrac{x}{\ln x}$, find $f'(e)$.

30. If $f(x) = x^2 \ln x$, find $f'(1)$.

31–32 □ Find an equation of the tangent line to the curve at the given point.

31. $y = \ln \ln x, \quad (e, 0)$

32. $y = \ln(x^2 + 1), \quad (1, \ln 2)$

33. If $f(x) = \sin x + \ln x$, find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

34. Find equations of the tangent lines to the curve $y = (\ln x)/x$ at the points $(1, 0)$ and $(e, 1/e)$. Illustrate by graphing the curve and its tangent lines.

35–46 □ Use logarithmic differentiation to find the derivative of the function.

35. $y = (2x + 1)^5 (x^4 - 3)^6$

36. $y = \sqrt{x}\, e^{x^2} (x^2 + 1)^{10}$

37. $y = \dfrac{\sin^2 x \tan^4 x}{(x^2 + 1)^2}$

38. $y = \sqrt[4]{\dfrac{x^2 + 1}{x^2 - 1}}$

39. $y = x^x$

40. $y = x^{1/x}$

41. $y = x^{\sin x}$

42. $y = (\sin x)^x$

43. $y = (\ln x)^x$

44. $y = x^{\ln x}$

45. $y = x^{e^x}$

46. $y = (\ln x)^{\cos x}$

47. Find y' if $y = \ln(x^2 + y^2)$.

48. Find y' if $x^y = y^x$.

49. Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x - 1)$.

50. Find $\dfrac{d^9}{dx^9}(x^8 \ln x)$.

51. Use the definition of derivative to prove that

$$\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1$$

52. Show that $\displaystyle\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for any $x > 0$.

3.9 Hyperbolic Functions

Certain combinations of the exponential functions e^x and e^{-x} arise so frequently in mathematics and its applications that they deserve to be given special names. In many ways they are analogous to the trigonometric functions, and they have the same relationship to the hyperbola that the trigonometric functions have to the circle. For this reason they are collectively called **hyperbolic functions** and individually called **hyperbolic sine**, **hyperbolic cosine**, and so on.

Definition of the Hyperbolic Functions

$$\sinh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

The graphs of hyperbolic sine and cosine can be sketched using graphical addition as in Figures 1 and 2.