MST 10010: Calculus I

Exercise Set 3

Do only the following problems:

Section 2.7 (pages 154-155): 17, 19

Section 2.8 (pages 161-162): 7, 9 a), 13, 15, 17, 25.

Section 2.9 (pages 172-173): 19, 21, 23, 25, 27, 29 a), 39.

Section 3.1 (pages 189): 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 43, 45.

Section 3.2 (page 195): 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27 a), 29 a), 30 a), 33, 34.

Section 3.4 (pages 213-214): 1, 3, 5, 9, 11, 13, 21, 23, 25 a), 28 a), 29, 35, 37, 39, 43.

17. If a ball is thrown into the air with a velocity of 40 ft/s, its height (in feet) after t seconds is given by $y = 40t - 16t^2$. Find the velocity when $t = 2$.

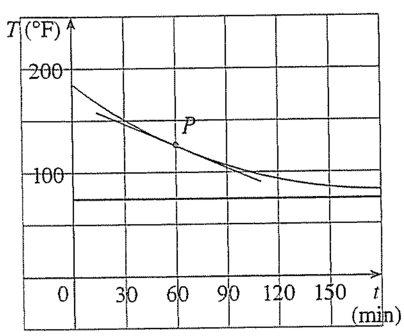
18. If an arrow is shot upward on the moon with a velocity of 58 m/s, its height (in meters) after t seconds is given by $H = 58t - 0.83t^2$.
    (a) Find the velocity of the arrow after one second.
    (b) Find the velocity of the arrow when $t = a$.
    (c) When will the arrow hit the moon?
    (d) With what velocity will the arrow hit the moon?

19. The displacement (in meters) of a particle moving in a straight line is given by the equation of motion $s = 4t^3 + 6t + 2$, where t is measured in seconds. Find the velocity of the particle at times $t = a$, $t = 1$, $t = 2$, and $t = 3$.

20. The displacement (in meters) of a particle moving in a straight line is given by $s = t^2 - 8t + 18$, where t is measured in seconds.
    (a) Find the average velocities over the following time intervals:
        (i) $[3, 4]$ (ii) $[3.5, 4]$
        (iii) $[4, 5]$ (iv) $[4, 4.5]$
    (b) Find the instantaneous velocity when $t = 4$.
    (c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities in part (a) and the tangent line whose slope is the instantaneous velocity in part (b).

21. A warm can of soda is placed in a cold refrigerator. Sketch the graph of the temperature of the soda as a function of time. Is the initial rate of change of temperature greater or less than the rate of change after an hour?

22. A roast turkey is taken from an oven when its temperature has reached 185 °F and is placed on a table in a room where the temperature is 75 °F. The graph shows how the temperature of the turkey decreases and eventually approaches room temperature. (In Section 9.4 we will be able to use Newton's Law of Cooling to find an equation for T as a function of time.) By measuring the slope of the tangent, estimate the rate of change of the temperature after an hour.



23. (a) Use the data in Example 5 to find the average rate of change of temperature with respect to time
        (i) from 8 P.M. to 11 P.M.
        (ii) from 8 P.M. to 10 P.M.
        (iii) from 8 P.M. to 9 P.M.
    (b) Estimate the instantaneous rate of change of T with respect to time at 8 P.M. by measuring the slope of a tangent.

24. The population P (in thousands) of the city of San Jose, California, from 1991 to 1997 is given in the table.

Year	1991	1993	1995	1997
P	793	820	839	874

    (a) Find the average rate of growth
        (i) from 1991 to 1995
        (ii) from 1993 to 1995
        (iii) from 1995 to 1997
        In each case, include the units.
    (b) Estimate the instantaneous rate of growth in 1995 by taking the average of two average rates of change. What are its units?
    (c) Estimate the instantaneous rate of growth in 1995 by measuring the slope of a tangent.

25. The cost (in dollars) of producing x units of a certain commodity is $C(x) = 5000 + 10x + 0.05x^2$.
    (a) Find the average rate of change of C with respect to x when the production level is changed
        (i) from $x = 100$ to $x = 105$
        (ii) from $x = 100$ to $x = 101$
    (b) Find the instantaneous rate of change of C with respect to x when $x = 100$. (This is called the *marginal cost*. Its significance will be explained in Section 3.3.)

26. If a cylindrical tank holds 100,000 gallons of water, which can be drained from the bottom of the tank in an hour, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V(t) = 100{,}000\left(1 - \frac{t}{60}\right)^2 \qquad 0 \leq t \leq 60$$

Find the rate at which the water is flowing out of the tank (the instantaneous rate of change of V with respect to t) as a function of t. What are its units? For times $t = 0$, 10, 20, 30, 40, 50, and 60 min, find the flow rate and the amount of water remaining in the tank. Summarize your findings in a sentence or two. At what time is the flow rate the greatest? The least?

So we compute and tabulate values of the difference quotient (the average rates of change) as follows:

t	$\dfrac{P(t) - P(1992)}{t - 1992}$
1988	2,625,750
1990	2,781,000
1994	2,645,000
1996	2,544,250

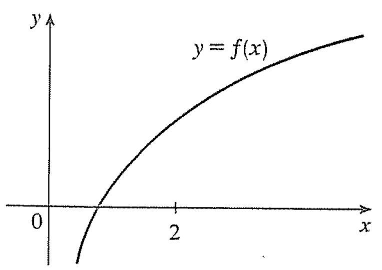
□ Another method is to plot the population function and estimate the slope of the tangent line when $t = 1992$. (See Example 5 in Section 2.7.)

From this table we see that $P'(1992)$ lies somewhere between 2,781,000 and 2,645,000. We estimate that the rate of increase of the population of the United States in 1992 was the average of these two numbers, namely

$$P'(1992) \approx 2.7 \text{ million people/year}$$

2.8 Exercises

1. On the given graph of f, mark lengths that represent $f(2)$, $f(2 + h)$, $f(2 + h) - f(2)$, and h. (Choose $h > 0$.) What line has slope $\dfrac{f(2 + h) - f(2)}{h}$?

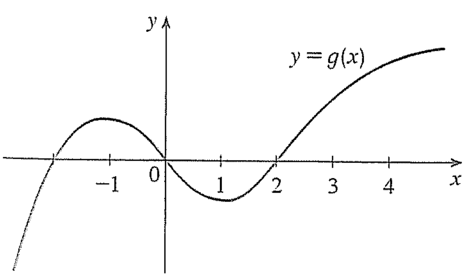


2. For the function f whose graph is shown in Exercise 1, arrange the following numbers in increasing order and explain your reasoning:

$$0 \qquad f'(2) \qquad f(3) - f(2) \qquad \tfrac{1}{2}[f(4) - f(2)]$$

3. For the function g whose graph is given, arrange the following numbers in increasing order and explain your reasoning:

$$0 \qquad g'(-2) \qquad g'(0) \qquad g'(2) \qquad g'(4)$$



4. If the tangent line to $y = f(x)$ at (4, 3) passes through the poir (0, 2), find $f(4)$ and $f'(4)$.

5. Sketch the graph of a function f for which $f(0) = 0$, $f'(0) = $ $f'(1) = 0$, and $f'(2) = -1$.

6. Sketch the graph of a function g for which $g(0) = 0$, $g'(0) = 3$ $g'(1) = 0$, and $g'(2) = 1$.

7. If $f(x) = 3x^2 - 5x$, find $f'(2)$ and use it to find an equation of the tangent line to the parabola $y = 3x^2 - 5x$ at the point (2, 2).

8. If $g(x) = 1 - x^3$, find $g'(0)$ and use it to find an equation of th tangent line to the curve $y = 1 - x^3$ at the point (0, 1).

9. (a) If $F(x) = x^3 - 5x + 1$, find $F'(1)$ and use it to find an equation of the tangent line to the curve $y = x^3 - 5x + 1$ at the point $(1, -3)$.
   (b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

10. (a) If $G(x) = x/(1 + 2x)$, find $G'(a)$ and use it to find an equation of the tangent line to the curve $y = x/(1 + 2x)$ at the point $\left(-\frac{1}{4}, -\frac{1}{2}\right)$.
   (b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

11. Let $f(x) = 3^x$. Estimate the value of $f'(1)$ in two ways:
   (a) By using Definition 2 and taking successively smaller values of h.
   (b) By zooming in on the graph of $y = 3^x$ and estimating the slope.

12. Let $g(x) = \tan x$. Estimate the value of $g'(\pi/4)$ in two ways:
   (a) By using Definition 2 and taking successively smaller values of h.

(b) By zooming in on the graph of $y = \tan x$ and estimating the slope.

13–18 □ Find $f'(a)$.

13. $f(x) = 1 + x - 2x^2$

14. $f(x) = x^3 + 3x$

15. $f(x) = \dfrac{x}{2x - 1}$

16. $f(x) = \dfrac{x}{x^2 - 1}$

17. $f(x) = \dfrac{2}{\sqrt{3 - x}}$

18. $f(x) = \sqrt{3x + 1}$

19–24 □ Each limit represents the derivative of some function f at some number a. State f and a in each case.

19. $\displaystyle\lim_{h \to 0} \frac{\sqrt{1 + h} - 1}{h}$

20. $\displaystyle\lim_{h \to 0} \frac{(2 + h)^3 - 8}{h}$

21. $\displaystyle\lim_{x \to 1} \frac{x^9 - 1}{x - 1}$

22. $\displaystyle\lim_{x \to 3\pi} \frac{\cos x + 1}{x - 3\pi}$

23. $\displaystyle\lim_{t \to 0} \frac{\sin\left(\frac{\pi}{2} + t\right) - 1}{t}$

24. $\displaystyle\lim_{x \to 0} \frac{3^x - 1}{x}$

25–26 □ A particle moves along a straight line with equation of motion $s = f(t)$, where s is measured in meters and t in seconds. Find the velocity when $t = 2$.

25. $f(t) = t^2 - 6t - 5$

26. $f(t) = 2t^3 - t + 1$

27. The cost of producing x ounces of gold from a new gold mine is $C = f(x)$ dollars.
(a) What is the meaning of the derivative $f'(x)$? What are its units?
(b) What does the statement $f'(800) = 17$ mean?
(c) Do you think the values of $f'(x)$ will increase or decrease in the short term? What about the long term? Explain.

28. The number of bacteria after t hours in a controlled laboratory experiment is $n = f(t)$.
(a) What is the meaning of the derivative $f'(5)$? What are its units?
(b) Suppose there is an unlimited amount of space and nutrients for the bacteria. Which do you think is larger, $f'(5)$ or $f'(10)$? If the supply of nutrients is limited, would that affect your conclusion? Explain.

29. The fuel consumption (measured in gallons per hour) of a car traveling at a speed of v miles per hour is $c = f(v)$.
(a) What is the meaning of the derivative $f'(v)$? What are its units?
(b) Write a sentence (in layman's terms) that explains the meaning of the equation $f'(20) = -0.05$.

30. The quantity (in pounds) of a gourmet ground coffee that is sold by a coffee company at a price of p dollars per pound is $Q = f(p)$.
(a) What is the meaning of the derivative $f'(8)$? What are its units?
(b) Is $f'(8)$ positive or negative? Explain.

31. Let $C(t)$ be the amount of U.S. cash per capita in circulation at time t. The table, supplied by the Treasury Department, gives values of $C(t)$ as of June 30 of the specified year. Interpret and estimate the value of $C'(1980)$.

t	1960	1970	1980	1990
$C(t)$	\$177	\$265	\$571	\$1063

32. Life expectancy has improved dramatically in the 20th century. The table gives values of $E(t)$, the life expectancy at birth (in years) of a male born in the year t in the United States. Interpret and estimate the values of $E'(1910)$ and $E'(1950)$.

t	$E(t)$	t	$E(t)$
1900	48.3	1950	65.6
1910	51.1	1960	66.6
1920	55.2	1970	67.1
1930	57.4	1980	70.0
1940	62.5	1990	71.8

33–34 □ Determine whether or not $f'(0)$ exists.

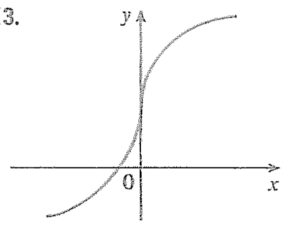
33. $f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

34. $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Writing Project — Early Methods for Finding Tangents

The first person to formulate explicitly the ideas of limits and derivatives was Sir Isaac Newton in the 1660s. But Newton acknowledged that "If I have seen farther than other men, it is because I have stood on the shoulders of giants." Two of those giants were Pierre Fermat (1601–1665) and Newton's teacher at Cambridge, Isaac Barrow (1630–1677). Newton was familiar with the methods that these men used to find tangent lines, and their methods played a role in Newton's eventual formulation of calculus.

13.



14–16 □ Make a careful sketch of the graph of f and below it sketch the graph of f' in the same manner as in Exercises 5–13. Can you guess a formula for $f'(x)$ from its graph?

14. $f(x) = \sin x$ **15.** $f(x) = e^x$

16. $f(x) = \ln x$

17. Let $f(x) = x^2$.
(a) Estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, and $f'(2)$ by using a graphing device to zoom in on the graph of f.
(b) Use symmetry to deduce the values of $f'(-\frac{1}{2})$, $f'(-1)$, and $f'(-2)$.
(c) Use the results from parts (a) and (b) to guess a formula for $f'(x)$.
(d) Use the definition of a derivative to prove that your guess in part (c) is correct.

18. Let $f(x) = x^3$.
(a) Estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, $f'(2)$, and $f'(3)$ by using a graphing device to zoom in on the graph of f.
(b) Use symmetry to deduce the values of $f'(-\frac{1}{2})$, $f'(-1)$, $f'(-2)$, and $f'(-3)$.
(c) Use the values from parts (a) and (b) to graph f'.
(d) Guess a formula for $f'(x)$.
(e) Use the definition of a derivative to prove that your guess in part (d) is correct.

19–27 □ Find the derivative of the given function using the definition of derivative. State the domain of the function and the domain of its derivative.

19. $f(x) = 5x + 3$ **20.** $f(x) = 5 - 4x + 3x^2$

21. $f(x) = x^3 - x^2 + 2x$ **22.** $f(x) = x + \sqrt{x}$

23. $g(x) = \sqrt{1 + 2x}$ **24.** $f(x) = \dfrac{x + 1}{x - 1}$

25. $G(x) = \dfrac{4 - 3x}{2 + x}$ **26.** $g(x) = \dfrac{1}{x^2}$

27. $f(x) = x^4$

28. (a) Sketch the graph of $f(x) = \sqrt{6 - x}$ by starting with the graph of $y = \sqrt{x}$ and using the transformations of Section 1.3.
(b) Use the graph from part (a) to sketch the graph of f'.
(c) Use the definition of a derivative to find $f'(x)$. What are the domains of f and f'?
(d) Use a graphing device to graph f' and compare with your sketch in part (b).

29. (a) If $f(x) = x - (2/x)$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

30. (a) If $f(t) = 6/(1 + t^2)$, find $f'(t)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

31. The unemployment rate $U(t)$ varies with time. The table (from the Bureau of Labor Statistics) gives the percentage of unemployed in the U. S. labor force from 1988 to 1997.

t	$U(t)$	t	$U(t)$
1988	5.5	1993	6.9
1989	5.3	1994	6.1
1990	5.6	1995	5.6
1991	6.8	1996	5.4
1992	7.5	1997	4.9

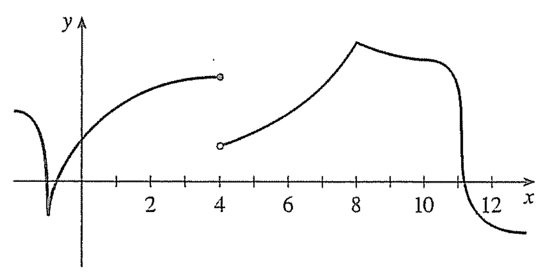
(a) What is the meaning of $U'(t)$? What are its units?
(b) Construct a table of values for $U'(t)$.

32. Let the smoking rate among high-school seniors at time t be $S(t)$. The table (from the Institute of Social Research, University of Michigan) gives the percentage of seniors who reported that they had smoked one or more cigarettes per day during the past 30 days.

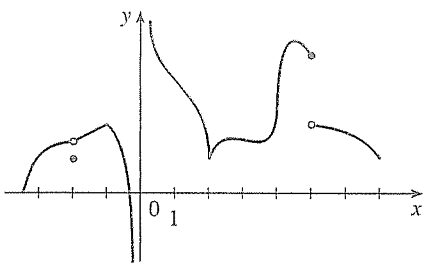
t	$S(t)$	t	$S(t)$
1978	27.5	1988	18.1
1980	21.4	1990	19.1
1982	21.0	1992	17.2
1984	18.7	1994	19.4
1986	18.7	1996	22.2

(a) What is the meaning of $S'(t)$? What are its units?
(b) Construct a table of values for $S'(t)$.
(c) Graph S and S'.
(d) How would it be possible to get more accurate values for $S'(t)$?

33. The graph of f is given. State, with reasons, the numbers at which f is not differentiable.



34. The graph of g is given.
(a) At what numbers is g discontinuous? Why?
(b) At what numbers is g not differentiable? Why?



35. Graph the function $f(x) = x + \sqrt{|x|}$. Zoom in repeatedly, first toward the point $(-1, 0)$ and then toward the origin. What is different about the behavior of f in the vicinity of these two points? What do you conclude about the differentiability of f?

36. Zoom in toward the points $(1, 0)$, $(0, 1)$, and $(-1, 0)$ on the graph of the function $g(x) = (x^2 - 1)^{2/3}$. What do you notice? Account for what you see in terms of the differentiability of g.

37. Let $f(x) = \sqrt[3]{x}$.
(a) If $a \neq 0$, use Equation 2.8.3 to find $f'(a)$.
(b) Show that $f'(0)$ does not exist.
(c) Show that $y = \sqrt[3]{x}$ has a vertical tangent line at $(0, 0)$. (Recall the shape of the graph of f. See Figure 13 in Section 1.2.)

38. (a) If $g(x) = x^{2/3}$, show that $g'(0)$ does not exist.
(b) If $a \neq 0$, find $g'(a)$.
(c) Show that $y = x^{2/3}$ has a vertical tangent line at $(0, 0)$.
(d) Illustrate part (c) by graphing $y = x^{2/3}$.

39. Show that the function $f(x) = |x - 6|$ is not differentiable at 6. Find a formula for f' and sketch its graph.

40. Where is the greatest integer function $f(x) = [\![x]\!]$ not differentiable? Find a formula for f' and sketch its graph.

41. (a) Sketch the graph of the function $f(x) = x|x|$.
(b) For what values of x is f differentiable?
(c) Find a formula for f'.

42. The **left-hand** and **right-hand derivatives** of f at a are defined by

$$f'_-(a) = \lim_{h \to 0^-} \frac{f(a + h) - f(a)}{h}$$

and

$$f'_+(a) = \lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h}$$

if these limits exist. Then $f'(a)$ exists if and only if these one-sided derivatives exist and are equal.
(a) Find $f'_-(4)$ and $f'_+(4)$ for the function

$$f(x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ 5 - x & \text{if } 0 < x < 4 \\ \dfrac{1}{5 - x} & \text{if } x \geqslant 4 \end{cases}$$

(b) Sketch the graph of f.
(c) Where is f discontinuous?
(d) Where is f not differentiable?

43. Recall that a function f is called *even* if $f(-x) = f(x)$ for all x in its domain and *odd* if $f(-x) = -f(x)$ for all such x. Prove each of the following.
(a) The derivative of an even function is an odd function.
(b) The derivative of an odd function is an even function.

44. When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running.
(a) Sketch a possible graph of T as a function of the time t that has elapsed since the faucet was turned on.
(b) Describe how the rate of change of T with respect to t varies as t increases.
(c) Sketch a graph of the derivative of T.

45. Let ℓ be the tangent line to the parabola $y = x^2$ at the point $(1, 1)$. The *angle of inclination* of ℓ is the angle ϕ that ℓ makes with the positive direction of the x-axis. Calculate ϕ correct to the nearest degree.

3.1 Exercises

1. (a) How is the number e defined?
   (b) Use a calculator to estimate the values of the limits

$$\lim_{h \to 0} \frac{2.7^h - 1}{h} \quad \text{and} \quad \lim_{h \to 0} \frac{2.8^h - 1}{h}$$

correct to two decimal places. What can you conclude about the value of e?

2. (a) Sketch, by hand, the graph of the function $f(x) = e^x$, paying particular attention to how the graph crosses the y-axis. What fact allows you to do this?
   (b) What types of functions are $f(x) = e^x$ and $g(x) = x^e$? Compare the differentiation formulas for f and g.
   (c) Which of the two functions in part (b) grows more rapidly when x is large?

3–28 □ Differentiate the function.

3. $f(x) = 5x - 1$
4. $F(x) = -4x^{10}$
5. $f(x) = x^2 + 3x - 4$
6. $g(x) = 5x^8 - 2x^5 + 6$
7. $y = x^{-2/5}$
8. $y = 5e^x + 3$
9. $V(r) = \frac{4}{3}\pi r^3$
10. $R(t) = 5t^{-3/5}$
11. $Y(t) = 6t^{-9}$
12. $R(x) = \dfrac{\sqrt{10}}{x^7}$
13. $F(x) = (16x)^3$
14. $y = \sqrt[3]{x}$
15. $g(x) = x^2 + \dfrac{1}{x^2}$
16. $f(t) = \sqrt{t} - \dfrac{1}{\sqrt{t}}$
17. $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}}$
18. $y = \dfrac{x^2 - 2\sqrt{x}}{x}$
19. $y = 3x + 2e^x$
20. $y = \sqrt{x}\,(x - 1)$
21. $y = 4\pi^2$
22. $y = x^{4/3} - x^{2/3}$
23. $y = ax^2 + bx + c$
24. $y = A + \dfrac{B}{x} + \dfrac{C}{x^2}$
25. $y = x + \sqrt[5]{x^2}$
26. $u = \sqrt[3]{t^2} + 2\sqrt{t^3}$
27. $v = x\sqrt{x} + \dfrac{1}{x^2\sqrt{x}}$
28. $y = e^{x+1} + 1$

29–34 □ Find $f'(x)$. Compare the graphs of f and f' and use them to explain why your answer is reasonable.

29. $f(x) = 2x^2 - x^4$
30. $f(x) = 3x^5 - 20x^3 + 50x$
31. $f(x) = 3x^{15} - 5x^3 + 3$
32. $f(x) = x + \dfrac{1}{x}$
33. $f(x) = x - 3x^{1/3}$
34. $f(x) = x^2 + 2e^x$

35. (a) By zooming in on the graph of $f(x) = x^{2/5}$, estimate the value of $f'(2)$.
   (b) Use the Power Rule to find the exact value of $f'(2)$ and compare with your estimate in part (a).

36. (a) By zooming in on the graph of $f(x) = x^2 - 2e^x$, estimate the value of $f'(1)$.
   (b) Find the exact value of $f'(1)$ and compare with your estimate in part (a).

37–40 □ Find an equation of the tangent line to the curve at the given point. Illustrate by graphing the curve and the tangent line on the same screen.

37. $y = x + \dfrac{4}{x}$, $(2, 4)$
38. $y = x^{5/2}$, $(4, 32)$
39. $y = x + \sqrt{x}$, $(1, 2)$
40. $y = x^2 + 2e^x$, $(0, 2)$

41. (a) Use a graphing calculator or computer to graph the function $f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30$ in the viewing rectangle $[-3, 5]$ by $[-10, 50]$.
   (b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of f'. (See Example 1 in Section 2.9.)
   (c) Calculate $f'(x)$ and use this expression, with a graphing device, to graph f'. Compare with your sketch in part (b).

42. (a) Use a graphing calculator or computer to graph the function $g(x) = e^x - 3x^2$ in the viewing rectangle $[-1, 4]$ by $[-8, 8]$.
   (b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of g'. (See Example 1 in Section 2.9.)
   (c) Calculate $g'(x)$ and use this expression, with a graphing device, to graph g'. Compare with your sketch in part (b).

43. Find the points on the curve $y = x^3 - x^2 - x + 1$ where the tangent is horizontal.

44. For what values of x does the graph of $f(x) = 2x^3 - 3x^2 - 6x + 87$ have a horizontal tangent?

45. Show that the curve $y = 6x^3 + 5x - 3$ has no tangent line with slope 4.

46. At what point on the curve $y = 1 + 2e^x - 3x$ is the tangent line parallel to the line $3x - y = 5$? Illustrate by graphing the curve and both lines.

47. Draw a diagram to show that there are two tangent lines to the parabola $y = x^2$ that pass through the point $(0, -4)$. Find the coordinates of the points where these tangent lines intersect the parabola.

3.2 Exercises

1. Find the derivative of $y = (x^2 + 1)(x^3 + 1)$ in two ways: by using the Product Rule and by performing the multiplication first. Do your answers agree?

2. Find the derivative of the function
$$F(x) = \frac{x - 3x\sqrt{x}}{\sqrt{x}}$$
in two ways: by using the Quotient Rule and by simplifying first. Show that your answers are equivalent. Which method do you prefer?

3–22 □ Differentiate.

Resources / Module 4 / Polynomial Models / Problems and Tests

3. $f(x) = x^2e^x$
4. $g(x) = \sqrt{x}\, e^x$
5. $y = \dfrac{e^x}{x^2}$
6. $y = \dfrac{e^x}{1 + x}$
7. $h(x) = \dfrac{x + 2}{x - 1}$
8. $f(u) = \dfrac{1 - u^2}{1 + u^2}$
9. $G(s) = (s^2 + s + 1)(s^2 + 2)$
10. $g(x) = (1 + \sqrt{x})(x - x^3)$
11. $H(x) = (x^3 - x + 1)(x^{-2} + 2x^{-3})$
12. $H(t) = e^t(1 + 3t^2 + 5t^4)$
13. $y = \dfrac{3t - 7}{t^2 + 5t - 4}$
14. $y = \dfrac{4t + 5}{2 - 3t}$
15. $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}}$
16. $y = \dfrac{\sqrt{x} - 1}{\sqrt{x} + 1}$
17. $y = (r^2 - 2r)e^r$
18. $y = \dfrac{u^2 - u - 2}{u + 1}$
19. $y = \dfrac{1}{x^4 + x^2 + 1}$
20. $y = \dfrac{e^x}{x + e^x}$
21. $f(x) = \dfrac{x}{x + \dfrac{c}{x}}$
22. $f(x) = \dfrac{ax + b}{cx + d}$

23–26 □ Find an equation of the tangent line to the curve at the given point.

23. $y = \dfrac{2x}{x + 1}$, $(1, 1)$
24. $y = \dfrac{\sqrt{x}}{x + 1}$, $(4, 0.4)$
25. $y = 2xe^x$, $(0, 0)$
26. $y = \dfrac{e^x}{x}$, $(1, e)$

27. (a) The curve $y = 1/(1 + x^2)$ is called a **witch of Maria Agnesi**. Find an equation of the tangent line to this curve at the point $\left(-1, \frac{1}{2}\right)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

28. (a) The curve $y = x/(1 + x^2)$ is called a **serpentine**. Find an equation of the tangent line to this curve at the point $(3, 0.3)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

29. (a) If $f(x) = e^x/x^3$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

30. (a) If $f(x) = x/(x^2 - 1)$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

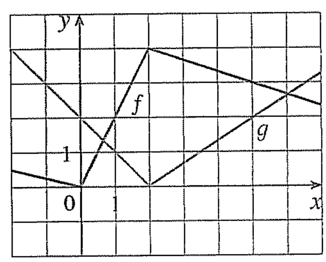
31. Suppose that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$. Find the values of (a) $(fg)'(5)$, (b) $(f/g)'(5)$, and (c) $(g/f)'(5)$.

32. If $f(3) = 4$, $g(3) = 2$, $f'(3) = -6$, and $g'(3) = 5$, find the following numbers:
(a) $(f + g)'(3)$
(b) $(fg)'(3)$
(c) $(f/g)'(3)$
(d) $\left(\dfrac{f}{f - g}\right)'(3)$

33. If $f(x) = e^xg(x)$, where $g(0) = 2$ and $g'(0) = 5$, find $f'(0)$.

34. If $h(2) = 4$ and $h'(2) = -3$, find
$$\left.\frac{d}{dx}\left(\frac{h(x)}{x}\right)\right|_{x=2}$$

35. If f and g are the functions whose graphs are shown, let $u(x) = f(x)g(x)$ and $v(x) = f(x)/g(x)$.
(a) Find $u'(1)$.
(b) Find $v'(5)$.



36. If f is a differentiable function, find an expression for the derivative of each of the following functions.
(a) $y = x^2f(x)$
(b) $y = \dfrac{f(x)}{x^2}$
(c) $y = \dfrac{x^2}{f(x)}$
(d) $y = \dfrac{1 + xf(x)}{\sqrt{x}}$

37. In this exercise we estimate the rate at which the total persona income is rising in the Miami–Ft. Lauderdale metropolitan area. In July, 1993, the population of this area was 3,354,000, and the population was increasing at roughly 45,000 people pe year. The average annual income was $21,107 per capita, and this average was increasing at about $1900 per year (well above the national average of about $660 yearly). Use the

SOLUTION In order to apply Equation 2, we first rewrite the function by multiplying and dividing by 7:

Note that $\sin 7x \neq 7 \sin x$.

$$\frac{\sin 7x}{4x} = \frac{7}{4}\left(\frac{\sin 7x}{7x}\right)$$

Notice that as $x \to 0$, we have $7x \to 0$, and so, by Equation 2 with $\theta = 7x$,

$$\lim_{x\to 0} \frac{\sin 7x}{7x} = \lim_{7x\to 0} \frac{\sin(7x)}{7x} = 1$$

Thus

$$\lim_{x\to 0} \frac{\sin 7x}{4x} = \lim_{x\to 0} \frac{7}{4}\left(\frac{\sin 7x}{7x}\right)$$

$$= \frac{7}{4} \lim_{x\to 0} \frac{\sin 7x}{7x} = \frac{7}{4} \cdot 1 = \frac{7}{4}$$

EXAMPLE 5 □ Calculate $\lim_{x\to 0} x \cot x$.

SOLUTION Here we divide numerator and denominator by x:

$$\lim_{x\to 0} x \cot x = \lim_{x\to 0} \frac{x \cos x}{\sin x}$$

$$= \lim_{x\to 0} \frac{\cos x}{\dfrac{\sin x}{x}} = \frac{\lim_{x\to 0} \cos x}{\lim_{x\to 0} \dfrac{\sin x}{x}}$$

$$= \frac{\cos 0}{1} \qquad \text{(by the continuity of cosine and Equation 2)}$$

$$= 1$$

3.4 Exercises

1–16 □ Differentiate.

Resources / Module 4 / Trigonometric Models / Derivatives of Trig Functions and Quiz

1. $f(x) = x - 3 \sin x$
2. $f(x) = x \sin x$
3. $y = \sin x + \cos x$
4. $y = \cos x - 2 \tan x$
5. $g(t) = t^3 \cos t$
6. $g(t) = 4 \sec t + \tan t$
7. $h(\theta) = \csc \theta + e^\theta \cot \theta$
8. $y = e^x \sin x$
9. $y = \dfrac{\tan x}{x}$
10. $y = \dfrac{\sin x}{1 + \cos x}$
11. $y = \dfrac{x}{\sin x + \cos x}$
12. $y = \dfrac{\tan x - 1}{\sec x}$
13. $y = \dfrac{\sin x}{x^2}$
14. $y = \tan \theta (\sin \theta + \cos \theta)$
15. $y = \csc x \cot x$
16. $y = x \sin x \cos x$

17. Prove that $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$.
18. Prove that $\dfrac{d}{dx}(\sec x) = \sec x \tan x$.
19. Prove that $\dfrac{d}{dx}(\cot x) = -\csc^2 x$.
20. Prove, using the definition of derivative, that if $f(x) = \cos x$, then $f'(x) = -\sin x$.

21–24 □ Find an equation of the tangent line to the given curve at the specified point.

21. $y = \tan x$, $(\pi/4, 1)$
22. $y = 2 \sin x$, $(\pi/6, 1)$
23. $y = x + \cos x$, $(0, 1)$
24. $y = \dfrac{1}{\sin x + \cos x}$, $(0, 1)$

25. (a) Find an equation of the tangent line to the curve $y = x \cos x$ at the point $(\pi, -\pi)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

26. (a) Find an equation of the tangent line to the curve $y = \sec x - 2 \cos x$ at the point $(\pi/3, 1)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

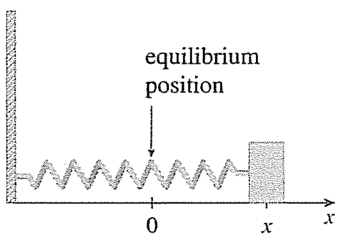
27. (a) If $f(x) = 2x + \cot x$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by graphing both f and f' for $0 < x < \pi$.

28. (a) If $f(x) = \sqrt{x} \sin x$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by graphing both f and f' for $0 \leqslant x \leqslant 2\pi$.

29. For what values of x does the graph of $f(x) = x + 2 \sin x$ have a horizontal tangent?

30. Find the points on the curve $y = (\cos x)/(2 + \sin x)$ at which the tangent is horizontal.

31. A mass on a spring vibrates horizontally on a smooth level surface (see the figure). Its equation of motion is $x(t) = 8 \sin t$, where t is in seconds and x in centimeters.
(a) Find the velocity at time t.
(b) Find the position and velocity of the mass at time $t = 2\pi/3$. In what direction is it moving at that time?



32. An elastic band is hung on a hook and a mass is hung on the lower end of the band. When the mass is pulled downward and then released, it vibrates vertically. The equation of motion is $s = 2 \cos t + 3 \sin t$, $t \geqslant 0$, where s is measured in centimeters and t in seconds. (We take the positive direction to be downward.)
(a) Find the velocity at time t.
(b) Graph the velocity and position functions.
(c) When does the mass pass through the equilibrium position for the first time?
(d) How far from its equilibrium position does the mass travel?
(e) When is the speed the greatest?

33. A ladder 10 ft long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall and let x be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does x change with respect to θ when $\theta = \pi/3$?

34. An object with weight W is dragged along a horizontal plan by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin \theta + \cos \theta}$$

where μ is a constant called the *coefficient of friction*.
(a) Find the rate of change of F with respect to θ.
(b) When is this rate of change equal to 0?
(c) If $W = 50$ lb and $\mu = 0.6$, draw the graph of F as a function of θ and use it to locate the value of θ for whic $dF/d\theta = 0$. Is the value consistent with your answer to part (b)?

35–44 □ Find the limit.

35. $\displaystyle\lim_{t \to 0} \frac{\sin 5t}{t}$

36. $\displaystyle\lim_{t \to 0} \frac{\sin 8t}{\sin 9t}$

37. $\displaystyle\lim_{\theta \to 0} \frac{\sin(\cos \theta)}{\sec \theta}$

38. $\displaystyle\lim_{\theta \to 0} \frac{\cos \theta - 1}{\sin \theta}$

39. $\displaystyle\lim_{\theta \to 0} \frac{\sin^2\theta}{\theta}$

40. $\displaystyle\lim_{x \to 0} \frac{\tan x}{4x}$

41. $\displaystyle\lim_{x \to 0} \frac{\cot 2x}{\csc x}$

42. $\displaystyle\lim_{x \to \pi/4} \frac{\sin x - \cos x}{\cos 2x}$

43. $\displaystyle\lim_{\theta \to 0} \frac{\sin \theta}{\theta + \tan \theta}$

44. $\displaystyle\lim_{x \to 1} \frac{\sin(x - 1)}{x^2 + x - 2}$

45. Differentiate each trigonometric identity to obtain a new (or familiar) identity.
(a) $\tan x = \dfrac{\sin x}{\cos x}$
(b) $\sec x = \dfrac{1}{\cos x}$
(c) $\sin x + \cos x = \dfrac{1 + \cot x}{\csc x}$

46. A semicircle with diameter PQ sits on an isosceles triangle PQR to form a region shaped like an ice-cream cone, as sho