MST 10010: Calculus I

Exercise Set 2

Do only the following problems:

Section 2.5 (pages 131-132): 1, 3, 5, 11, 13, 17, 19, 21, 23, 31, 35, 37, 39, 43, 45.

Section 2.6 (pages 144-145): 3, 5, 7, 11, 15, 17, 19, 21, 23, 25, 27, 29, 31, 35 and 37 (just find the horizontal and vertical asymptotes. Do not graph).

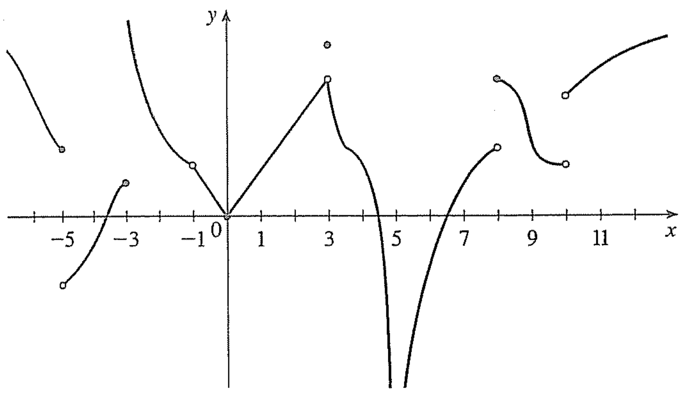
Section 2.7 (pages 144-145): 1, 7, 9, 11, 13 a) and b).

2.5 Exercises

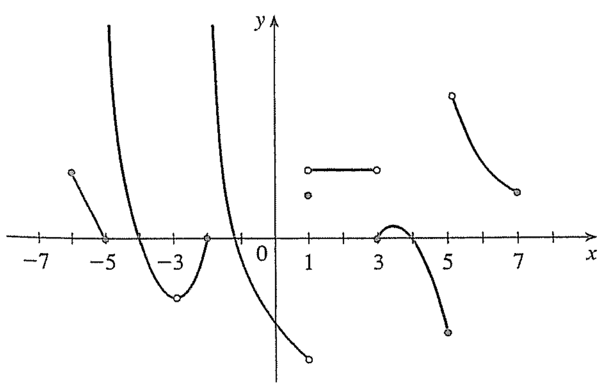
1. Write an equation that expresses the fact that a function f is continuous at the number 4.

2. If f is continuous on $(-\infty, \infty)$, what can you say about its graph?

3. (a) From the graph of f, state the numbers at which f is discontinuous and explain why.
   (b) For each of the numbers stated in part (a), determine whether f is continuous from the right, or from the left, or neither.



4. From the graph of g, state the intervals on which g is continuous.



5. Sketch the graph of a function that is continuous everywhere except at $x = 3$ and is continuous from the left at 3.

6. Sketch the graph of a function that has a jump discontinuity at $x = 2$ and a removable discontinuity at $x = 4$, but is continuous elsewhere.

7. A parking lot charges \$3 for the first hour (or part of an hour) and \$2 for each succeeding hour (or part), up to a daily maximum of \$10.
   (a) Sketch a graph of the cost of parking at this lot as a function of the time parked there.
   (b) Discuss the discontinuities of this function and their significance to someone who parks in the lot.

8. Explain why each function is continuous or discontinuous.
   (a) The temperature at a specific location as a function of time
   (b) The temperature at a specific time as a function of the distance due west from New York City
   (c) The altitude above sea level as a function of the distance due west from New York City
   (d) The cost of a taxi ride as a function of the distance traveled
   (e) The current in the circuit for the lights in a room as a function of time

9. If f and g are continuous functions with $f(3) = 5$ and $\lim_{x \to 3}[2f(x) - g(x)] = 4$, find $g(3)$.

10–12 □ Use the definition of continuity and the properties of limits to show that the function is continuous at the given number.

10. $f(x) = x^2 + \sqrt{7 - x}, \quad a = 4$

11. $f(x) = (x + 2x^3)^4, \quad a = -1$

12. $g(x) = \dfrac{x+1}{2x^2 - 1}, \quad a = 4$

13–14 □ Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

13. $f(x) = x\sqrt{16 - x^2}, \quad [-4, 4]$

14. $F(x) = \dfrac{x+1}{x-3}, \quad (-\infty, 3)$

15–20 □ Explain why the function is discontinuous at the given number. Sketch the graph of the function.

15. $f(x) = \ln|x - 2| \qquad a = 2$

16. $f(x) = \begin{cases} \dfrac{1}{x-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases} \qquad a = 1$

17. $f(x) = \dfrac{x^2 - 1}{x + 1} \qquad a = -1$

18. $f(x) = \begin{cases} \dfrac{x^2 - 1}{x + 1} & \text{if } x \neq -1 \\ 6 & \text{if } x = -1 \end{cases} \qquad a = -1$

19. $f(x) = \begin{cases} \dfrac{x^2 - 2x - 8}{x - 4} & \text{if } x \neq 4 \\ 3 & \text{if } x = 4 \end{cases} \qquad a = 4$

20. $f(x) = \begin{cases} 1 - x & \text{if } x \leqslant 2 \\ x^2 - 2x & \text{if } x > 2 \end{cases} \qquad a = 2$

21–28 □ Explain, using Theorems 4, 5, 7, and 9, why the function is continuous at every number in its domain. State the domain.

21. $F(x) = \dfrac{x}{x^2 + 5x + 6}$

22. $f(t) = 2t + \sqrt{25 - t^2}$

23. $h(x) = \sqrt[5]{x - 1}\,(x^2 - 2)$

24. $h(x) = \dfrac{\sin x}{x + 1}$

25. $f(x) = e^x \sin 5x$

26. $F(x) = \sin^{-1}(x^2 - 1)$

27. $G(t) = \ln(t^4 - 1)$

28. $H(x) = \cos(e^{\sqrt{x}})$

29–30 □ Locate the discontinuities of the function and illustrate by graphing.

29. $y = \dfrac{1}{1 + e^{1/x}}$

30. $y = \ln(\tan^2 x)$

31–34 □ Use continuity to evaluate the limit.

31. $\displaystyle\lim_{x \to 4} \frac{5 + \sqrt{x}}{\sqrt{5 + x}}$

32. $\displaystyle\lim_{x \to \pi} \sin(x + \sin x)$

33. $\displaystyle\lim_{x \to 1} e^{x^2 - x}$

34. $\displaystyle\lim_{x \to 2} \arctan\left(\frac{x^2 - 4}{3x^2 - 6x}\right)$

35. Let

$$f(x) = \begin{cases} x - 1 & \text{for } x < 3 \\ 5 - x & \text{for } x \geqslant 3 \end{cases}$$

Show that f is continuous on $(-\infty, \infty)$.

36–37 □ Find the points at which f is discontinuous. At which of these points is f continuous from the right, from the left, or neither? Sketch the graph of f.

36. $f(x) = \begin{cases} 2x + 1 & \text{if } x \leqslant -1 \\ 3x & \text{if } -1 < x < 1 \\ 2x - 1 & \text{if } x \geqslant 1 \end{cases}$

37. $f(x) = \begin{cases} (x - 1)^3 & \text{if } x < 0 \\ (x + 1)^3 & \text{if } x \geqslant 0 \end{cases}$

38. The gravitational force exerted by Earth on a unit mass at a distance r from the center of the planet is

$$F(r) = \begin{cases} \dfrac{GMr}{R^3} & \text{if } r < R \\[2ex] \dfrac{GM}{r^2} & \text{if } r \geqslant R \end{cases}$$

where M is the mass of Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r?

39. For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx + 1 & \text{if } x \leqslant 3 \\ cx^2 - 1 & \text{if } x > 3 \end{cases}$$

40. Find the constant c that makes g continuous on $(-\infty, \infty)$.

$$g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4 \\ cx + 20 & \text{if } x \geqslant 4 \end{cases}$$

41. Which of the following functions f has a removable discontinuity at a? If the discontinuity is removable, find a function g that agrees with f for $x \neq a$ and is continuous on $\mathbb{R}$.

(a) $f(x) = \dfrac{x^2 - 2x - 8}{x + 2}, \quad a = -2$

(b) $f(x) = \dfrac{x - 7}{|x - 7|}, \quad a = 7$

(c) $f(x) = \dfrac{x^3 + 64}{x + 4}, \quad a = -4$

(d) $f(x) = \dfrac{3 - \sqrt{x}}{9 - x}, \quad a = 9$

42. Suppose that a function f is continuous on $[0, 1]$ except at 0.25 and that $f(0) = 1$ and $f(1) = 3$. Let $N = 2$. Sketch two possible graphs of f, one showing that f might not satisfy the conclusion of the Intermediate Value Theorem and one showing that f might still satisfy the conclusion of the Intermediate Value Theorem (even though it doesn't satisfy the hypothesis).

43. If $f(x) = x^3 - x^2 + x$, show that there is a number c such that $f(c) = 10$.

44. Use the Intermediate Value Theorem to prove that there is a positive number c such that $c^2 = 2$. (This proves the existence of the number $\sqrt{2}$.)

45–48 □ Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

45. $x^3 - 3x + 1 = 0, \quad (0, 1)$

46. $x^2 = \sqrt{x + 1}, \quad (1, 2)$

47. $\cos x = x, \quad (0, 1)$

48. $\ln x = e^{-x}, \quad (1, 2)$

49–50 □ (a) Prove that the equation has at least one real root.
(b) Use your calculator to find an interval of length 0.01 that contains a root.

49. $e^x = 2 - x$

50. $x^5 - x^2 + 2x + 3 = 0$

51–52 □ (a) Prove that the equation has at least one real root.
(b) Use your graphing device to find the root correct to three decimal places.

51. $x^5 - x^2 - 4 = 0$

52. $\sqrt{x - 5} = \dfrac{1}{x + 3}$

53. Prove that f is continuous at a if and only if

$$\lim_{h \to 0} f(a + h) = f(a)$$

54. To prove that sine is continuous we need to show that $\lim_{x \to a} \sin x = \sin a$ for every real number a. By Exercise 53

Therefore, by Definition 7,

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Figure 18 illustrates the proof by showing some values of ε and the corresponding values of N.

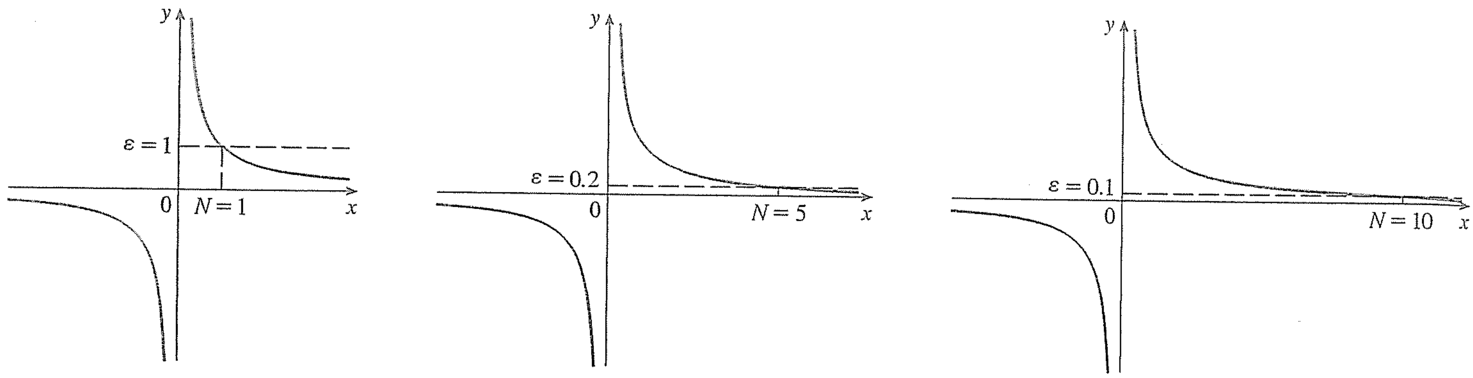


FIGURE 18

□

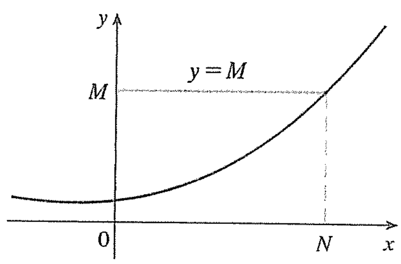


FIGURE 19
$\lim_{x \to \infty} f(x) = \infty$

Finally we note that an infinite limit at infinity can be defined as follows. The geometric illustration is given in Figure 19.

9 Definition Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

$$f(x) > M \qquad \text{whenever} \qquad x > N$$

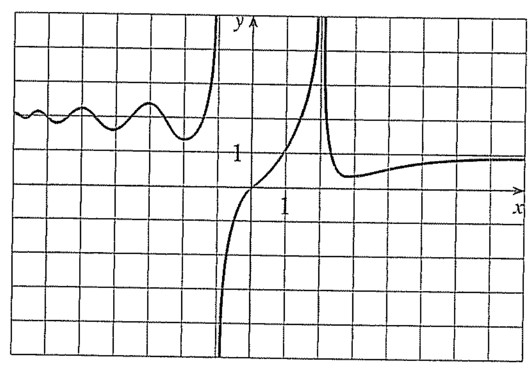
Similar definitions apply when the symbol ∞ is replaced by $-\infty$ (see Exercise 64).

2.6 Exercises

1. Explain in your own words the meaning of each of the following.
   (a) $\lim_{x \to \infty} f(x) = 5$ (b) $\lim_{x \to -\infty} f(x) = 3$

2. (a) Can the graph of $y = f(x)$ intersect a vertical asymptote? Can it intersect a horizontal asymptote? Illustrate by sketching graphs.
   (b) How many horizontal asymptotes can the graph of $y = f(x)$ have? Sketch graphs to illustrate the possibilities.

3. For the function f whose graph is given, state the following.
   (a) $\lim_{x \to 2} f(x)$ (b) $\lim_{x \to -1^-} f(x)$ (c) $\lim_{x \to -1^+} f(x)$
   (d) $\lim_{x \to \infty} f(x)$ (e) $\lim_{x \to -\infty} f(x)$
   (f) The equations of the asymptotes



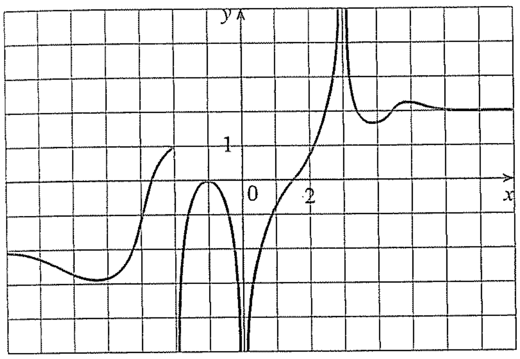
4. For the function g whose graph is given, state the following.

(a) $\lim_{x \to \infty} g(x)$ (b) $\lim_{x \to -\infty} g(x)$

(c) $\lim_{x \to 3} g(x)$ (d) $\lim_{x \to 0} g(x)$

(e) $\lim_{x \to -2^+} g(x)$

(f) The equations of the asymptotes



5–8 □ Sketch the graph of an example of a function f that satisfies all of the given conditions.

5. $f(0) = 0, \quad f(1) = 1, \quad \lim_{x \to \infty} f(x) = 0, \quad f$ is odd

6. $\lim_{x \to 0^+} f(x) = \infty, \quad \lim_{x \to 0^-} f(x) = -\infty, \quad \lim_{x \to \infty} f(x) = 1,$
$\lim_{x \to -\infty} f(x) = 1$

7. $\lim_{x \to 2} f(x) = -\infty, \quad \lim_{x \to \infty} f(x) = \infty, \quad \lim_{x \to -\infty} f(x) = 0,$
$\lim_{x \to 0^+} f(x) = \infty, \quad \lim_{x \to 0^-} f(x) = -\infty$

8. $\lim_{x \to -2} f(x) = \infty, \quad \lim_{x \to -\infty} f(x) = 3, \quad \lim_{x \to \infty} f(x) = -3$

9. Guess the value of the limit

$$\lim_{x \to \infty} \frac{x^2}{2^x}$$

by evaluating the function $f(x) = x^2/2^x$ for $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50$, and 100. Then use a graph of f to support your guess.

10. (a) Use a graph of

$$f(x) = \left(1 - \frac{2}{x}\right)^x$$

to estimate the value of $\lim_{x \to \infty} f(x)$ correct to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

11–14 □ Evaluate the limit and justify each step by indicating the appropriate properties of limits.

11. $\lim_{x \to \infty} \dfrac{x + 4}{x^2 - 2x + 5}$ **12.** $\lim_{t \to \infty} \dfrac{7t^3 + 4t}{2t^3 - t^2 + 3}$

13. $\lim_{x \to -\infty} \dfrac{(1 - x)(2 + x)}{(1 + 2x)(2 - 3x)}$ **14.** $\lim_{x \to \infty} \sqrt{\dfrac{2x^2 - 1}{x + 8x^2}}$

15–32 □ Find the limit.

15. $\lim_{r \to \infty} \dfrac{r^4 - r^2 + 1}{r^5 + r^3 - r}$ **16.** $\lim_{t \to -\infty} \dfrac{6t^2 + 5t}{(1 - t)(2t - 3)}$

17. $\lim_{x \to \infty} \dfrac{\sqrt{1 + 4x^2}}{4 + x}$ **18.** $\lim_{x \to -\infty} \dfrac{\sqrt{x^2 + 4x}}{4x + 1}$

19. $\lim_{x \to \infty} \dfrac{1 - \sqrt{x}}{1 + \sqrt{x}}$ **20.** $\lim_{x \to \infty} \left(\sqrt{x^2 + 3x + 1} - x\right)$

21. $\lim_{x \to \infty} \left(\sqrt{x^2 + 1} - \sqrt{x^2 - 1}\right)$ **22.** $\lim_{x \to -\infty} \left(x + \sqrt{x^2 + 2x}\right)$

23. $\lim_{x \to \infty} \left(\sqrt{9x^2 + x} - 3x\right)$ **24.** $\lim_{x \to \infty} \cos x$

25. $\lim_{x \to \infty} \sqrt{x}$ **26.** $\lim_{x \to -\infty} \sqrt[3]{x}$

27. $\lim_{x \to \infty} \left(x - \sqrt{x}\right)$ **28.** $\lim_{x \to \infty} \left(x + \sqrt{x}\right)$

29. $\lim_{x \to -\infty} (x^3 - 5x^2)$ **30.** $\lim_{x \to \infty} \tan^{-1}(x^2 - x^4)$

31. $\lim_{x \to \infty} \dfrac{x^7 - 1}{x^6 + 1}$ **32.** $\lim_{x \to \infty} e^{-x^2}$

33. (a) Estimate the value of

$$\lim_{x \to -\infty} \left(\sqrt{x^2 + x + 1} + x\right)$$

by graphing the function $f(x) = \sqrt{x^2 + x + 1} + x$.

(b) Use a table of values of $f(x)$ to guess the value of the limit.

(c) Prove that your guess is correct.

34. (a) Use a graph of

$$f(x) = \sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1}$$

to estimate the value of $\lim_{x \to \infty} f(x)$ to one decimal place.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal palces.

(c) Find the exact value of the limit.

35–40 □ Find the horizontal and vertical asymptotes of each curve. Check your work by graphing the curve and estimating the asymptotes.

35. $y = \dfrac{x}{x + 4}$ **36.** $y = \dfrac{x^2 + 4}{x^2 - 1}$

37. $y = \dfrac{x^3}{x^2 + 3x - 10}$ **38.** $y = \dfrac{x^3 + 1}{x^3 + x}$

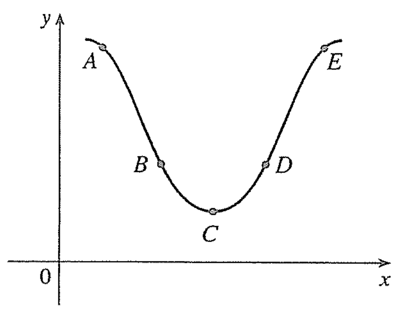
39. $h(x) = \dfrac{x}{\sqrt[4]{x^4 + 1}}$ **40.** $F(x) = \dfrac{x - 9}{\sqrt{4x^2 + 3x + 2}}$

2.7 Exercises

1. A curve has equation $y = f(x)$.
   (a) Write an expression for the slope of the secant line through the points $P(3, f(3))$ and $Q(x, f(x))$.
   (b) Write an expression for the slope of the tangent line at P.

2. Suppose an object moves with position function $s = f(t)$.
   (a) Write an expression for the average velocity of the object in the time interval from $t = a$ to $t = a + h$.
   (b) Write an expression for the instantaneous velocity at time $t = a$.

3. Consider the slope of the given curve at each of the five points shown. List these five slopes in decreasing order and explain your reasoning.



4. Graph the curve $y = e^x$ in the viewing rectangles $[-1, 1]$ by $[0, 2]$, $[-0.5, 0.5]$ by $[0.5, 1.5]$, and $[-0.1, 0.1]$ by $[0.9, 1.1]$. What do you notice about the curve as you zoom in toward the point $(0, 1)$?

5. (a) Find the slope of the tangent line to the parabola $y = x^2 + 2x$ at the point $(-3, 3)$
   (i) using Definition 1
   (ii) using Equation 2
   (b) Find the equation of the tangent line in part (a).
   (c) Graph the parabola and the tangent line. As a check on your work, zoom in toward the point $(-3, 3)$ until the parabola and the tangent line are indistinguishable.

6. (a) Find the slope of the tangent line to the curve $y = x^3$ at the point $(-1, -1)$
   (i) using Definition 1
   (ii) using Equation 2
   (b) Find the equation of the tangent line in part (a).
   (c) Graph the curve and the tangent line in successively smaller viewing rectangles centered at $(-1, -1)$ until the curve and the line appear to coincide.

7–10 □ Find an equation of the tangent line to the curve at the given point.

7. $y = 1 - 2x - 3x^2$, $(-2, -7)$
8. $y = 1/\sqrt{x}$, $(1, 1)$
9. $y = 1/x^2$, $\left(-2, \frac{1}{4}\right)$
10. $y = x/(1 - x)$, $(0, 0)$

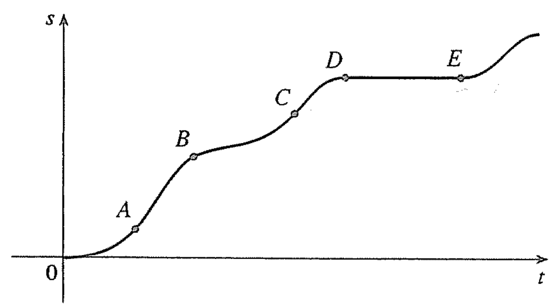
11. (a) Find the slope of the tangent to the curve $y = 2/(x + 3)$ at the point where $x = a$.
    (b) Find the slopes of the tangent lines at the points whose x-coordinates are (i) -1, (ii) 0, and (iii) 1.

12. (a) Find the slope of the tangent to the parabola $y = 1 + x + x^2$ at the point where $x = a$.
    (b) Find the slopes of the tangent lines at the points whose x-coordinates are (i) -1, (ii) $-\frac{1}{2}$, and (iii) 1.
    (c) Graph the curve and the three tangents on a common screen.

13. (a) Find the slope of the tangent to the curve $y = x^3 - 4x + 1$ at the point where $x = a$.
    (b) Find equations of the tangent lines at the points $(1, -2)$ and $(2, 1)$.
    (c) Graph the curve and both tangents on a common screen.

14. (a) Find the slope of the tangent to the curve $y = 1/\sqrt{5 - 2x}$ at the point where $x = a$.
    (b) Find equations of the tangent lines at the points $(2, 1)$ and $\left(-2, \frac{1}{3}\right)$.
    (c) Graph the curve and both tangents on a common screen.

15. The graph shows the position function of a car. Use the shape of the graph to explain your answers to the following questions.
    (a) What was the initial velocity of the car?
    (b) Was the car going faster at B or at C?
    (c) Was the car slowing down or speeding up at A, B, and C?
    (d) What happened between D and E?



16. Valerie is driving along a highway. Sketch the graph of the position function of her car if she drives in the following manner: At time $t = 0$, the car is at mile marker 15 and is traveling at a constant speed of 55 mi/h. She travels at this speed for exactly an hour. Then the car slows gradually over a 2-minute period as Valerie comes to a stop for dinner. Dinner lasts 26 min; then she restarts the car, gradually speeding up to 65 mi/h over a 2-minute period. She drives at a constant 65 mi/h for two hours and then over a 3-minute period gradually slows to a complete stop.

17. If a ball is thrown into the air with a velocity of 40 ft/s, its height (in feet) after t seconds is given by $y = 40t - 16t^2$. Find the velocity when $t = 2$.

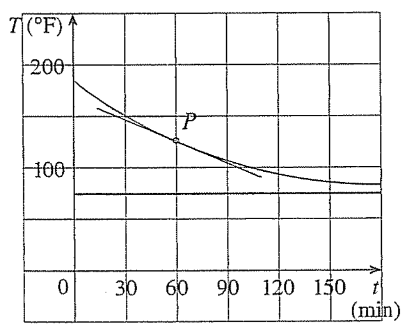
18. If an arrow is shot upward on the moon with a velocity of 58 m/s, its height (in meters) after t seconds is given by $H = 58t - 0.83t^2$.
   (a) Find the velocity of the arrow after one second.
   (b) Find the velocity of the arrow when $t = a$.
   (c) When will the arrow hit the moon?
   (d) With what velocity will the arrow hit the moon?

19. The displacement (in meters) of a particle moving in a straight line is given by the equation of motion $s = 4t^3 + 6t + 2$, where t is measured in seconds. Find the velocity of the particle at times $t = a$, $t = 1$, $t = 2$, and $t = 3$.

20. The displacement (in meters) of a particle moving in a straight line is given by $s = t^2 - 8t + 18$, where t is measured in seconds.
   (a) Find the average velocities over the following time intervals:
      (i) $[3, 4]$ (ii) $[3.5, 4]$
      (iii) $[4, 5]$ (iv) $[4, 4.5]$
   (b) Find the instantaneous velocity when $t = 4$.
   (c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities in part (a) and the tangent line whose slope is the instantaneous velocity in part (b).

21. A warm can of soda is placed in a cold refrigerator. Sketch the graph of the temperature of the soda as a function of time. Is the initial rate of change of temperature greater or less than the rate of change after an hour?

22. A roast turkey is taken from an oven when its temperature has reached 185 °F and is placed on a table in a room where the temperature is 75 °F. The graph shows how the temperature of the turkey decreases and eventually approaches room temperature. (In Section 9.4 we will be able to use Newton's Law of Cooling to find an equation for T as a function of time.) By



measuring the slope of the tangent, estimate the rate of change of the temperature after an hour.

23. (a) Use the data in Example 5 to find the average rate of change of temperature with respect to time
      (i) from 8 P.M. to 11 P.M.
      (ii) from 8 P.M. to 10 P.M.
      (iii) from 8 P.M. to 9 P.M.
   (b) Estimate the instantaneous rate of change of T with respect to time at 8 P.M. by measuring the slope of a tangent.

24. The population P (in thousands) of the city of San Jose, California, from 1991 to 1997 is given in the table.

Year	1991	1993	1995	1997
P	793	820	839	874

   (a) Find the average rate of growth
      (i) from 1991 to 1995
      (ii) from 1993 to 1995
      (iii) from 1995 to 1997
      In each case, include the units.
   (b) Estimate the instantaneous rate of growth in 1995 by taking the average of two average rates of change. What are its units?
   (c) Estimate the instantaneous rate of growth in 1995 by measuring the slope of a tangent.

25. The cost (in dollars) of producing x units of a certain commodity is $C(x) = 5000 + 10x + 0.05x^2$.
   (a) Find the average rate of change of C with respect to x when the production level is changed
      (i) from $x = 100$ to $x = 105$
      (ii) from $x = 100$ to $x = 101$
   (b) Find the instantaneous rate of change of C with respect to x when $x = 100$. (This is called the *marginal cost*. Its significance will be explained in Section 3.3.)

26. If a cylindrical tank holds 100,000 gallons of water, which can be drained from the bottom of the tank in an hour, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V(t) = 100{,}000\left(1 - \frac{t}{60}\right)^2 \qquad 0 \leqslant t \leqslant 60$$

Find the rate at which the water is flowing out of the tank (the instantaneous rate of change of V with respect to t) as a function of t. What are its units? For times $t = 0, 10, 20, 30, 40, 50$, and 60 min, find the flow rate and the amount of water remaining in the tank. Summarize your findings in a sentence or two. At what time is the flow rate the greatest? The least?