MST 10010: Calculus I

Exercise Set 1

Do only the following problems:

Section 1.1 (pages 22-24): 1 (a)-(f), 19, 21, 23, 29, 31, 33, 35, 37, 39, 47.

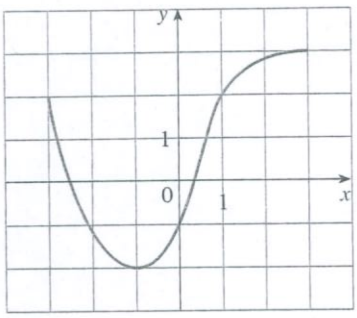
Section 1.3 (page 48): 31, 35, 37, 39, 41, 43, 45, 47, 49.

Section 2.2 (pages 99-101): 1, 3, 5, 7, 9, 13, 15, 17, 21, 23, 25.

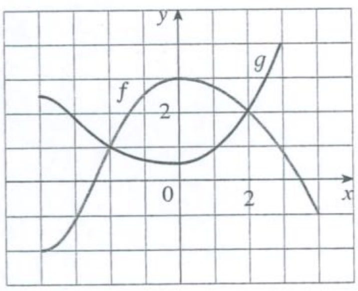
Section 2.3 (pages 109-111): 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 33, 35, 37, 39, 41.

1.1 Exercises

1. The graph of a function f is given.
   (a) State the value of $f(-1)$.
   (b) Estimate the value of $f(2)$.
   (c) For what values of x is $f(x) = 2$?
   (d) Estimate the values of x such that $f(x) = 0$.
   (e) State the domain and range of f.
   (f) On what interval is f increasing?



2. The graphs of f and g are given.
   (a) State the values of $f(-4)$ and $g(3)$.
   (b) For what values of x is $f(x) = g(x)$?
   (c) Estimate the solution of the equation $f(x) = -1$.
   (d) On what interval is f decreasing?
   (e) State the domain and range of f.
   (f) State the domain and range of g.

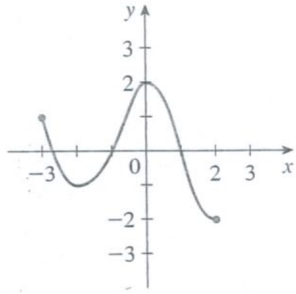


3. Figures 1, 11, and 12 were recorded by an instrument operated by the California Department of Mines and Geology at the University Hospital of the University of Southern California in Los Angeles. Use them to estimate the ranges of the vertical, north-south, and east-west ground acceleration functions at USC during the Northridge earthquake.

4. In this section we discussed examples of ordinary, everyday functions: population is a function of time, postage cost is a function of weight, water temperature is a function of time. Give three other examples of functions from everyday life that are described verbally. What can you say about the domain and range of each of your functions? If possible, sketch a rough graph of each function.

5–8 □ Determine whether the curve is the graph of a function of x. If it is, state the domain and range of the function.

5.



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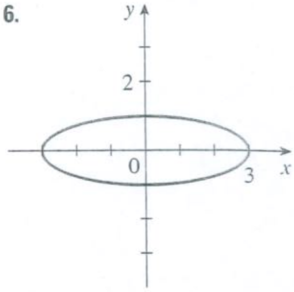

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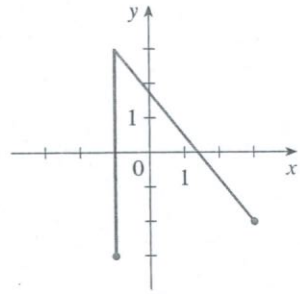

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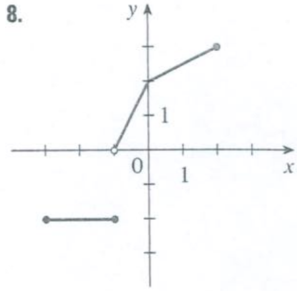
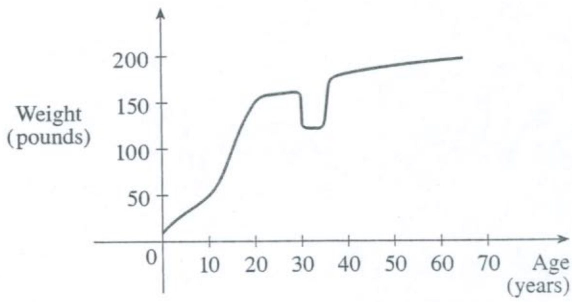
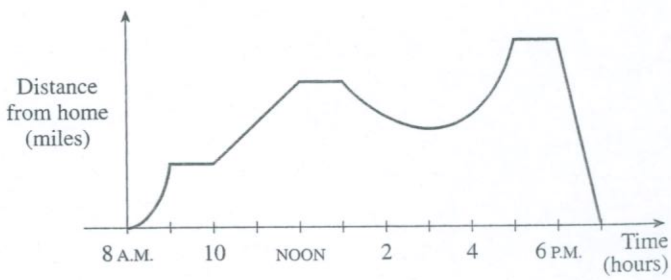


9. The graph shown gives the weight of a certain person as a function of age. Describe in words how this person's weight varies over time. What do you think happened when this person was 30 years old?



10. The graph shown gives a salesman's distance from his home as a function of time on a certain day. Describe in words what the graph indicates about his travels on this day.



11. You put some ice cubes in a glass, fill the glass with cold water, and then let the glass sit on a table. Describe how the temperature of the water changes as time passes. Then sketch a rough graph of the temperature of the water as a function of the elapsed time.

12. Sketch a rough graph of the number of hours of daylight as a function of the time of year.

13. Sketch a rough graph of the outdoor temperature as a function of time during a typical spring day.

14. You place a frozen pie in an oven and bake it for an hour. Then you take it out and let it cool before eating it. Describe how the temperature of the pie changes as time passes. Then sketch a rough graph of the temperature of the pie as a function of time.

15. A homeowner mows the lawn every Wednesday afternoon. Sketch a rough graph of the height of the grass as a function of time over the course of a four-week period.

16. An airplane flies from an airport and lands an hour later at another airport, 400 miles away. If t represents the time in minutes since the plane has left the terminal building, let $x(t)$ be the horizontal distance traveled and $y(t)$ be the altitude of the plane.
(a) Sketch a possible graph of $x(t)$.
(b) Sketch a possible graph of $y(t)$.
(c) Sketch a possible graph of the ground speed.
(d) Sketch a possible graph of the vertical velocity.

17. Temperature readings T (in °F) were recorded every two hours from midnight to noon in Atlanta, Georgia, on March 18, 1996. The time t was measured in hours from midnight.

t	0	2	4	6	8	10	12
T	58	57	53	50	51	57	61

(a) Use the readings to sketch a rough graph of T as a function of t.
(b) Use the graph to estimate the temperature at 11 A.M.

18. The population P (in thousands) of San Jose, California, from 1984 to 1994 is shown in the table. (Midyear estimates are given.)

t	1984	1986	1988	1990	1992	1994
P	695	716	733	782	800	817

(a) Draw a graph of P as a function of time.
(b) Use the graph to estimate the population in 1991.

19. If $f(x) = 2x^2 + 3x - 4$, find $f(0)$, $f(2)$, $f(\sqrt{2})$, $f(1 + \sqrt{2})$, $f(-x)$, $f(x + 1)$, $2f(x)$, and $f(2x)$.

20. A spherical balloon with radius r inches has volume $V(r) = \frac{4}{3}\pi r^3$. Find a function that represents the amount of air required to inflate the balloon from a radius of r inches to a radius of $r + 1$ inches.

21–22 □ Find $f(2 + h)$, $f(x + h)$, and $\dfrac{f(x + h) - f(x)}{h}$, where $h \neq 0$.

21. $f(x) = x - x^2$

22. $f(x) = \dfrac{x}{x + 1}$

23–27 □ Find the domain of the function.

23. $f(x) = \dfrac{x + 2}{x^2 - 1}$

24. $f(x) = \dfrac{x^4}{x^2 + x - 6}$

25. $g(x) = \sqrt[4]{x^2 - 6x}$

26. $h(x) = \sqrt[4]{7 - 3x}$

27. $f(t) = \sqrt[3]{t - 1}$

28. Find the domain and range and sketch the graph of the function $h(x) = \sqrt{4 - x^2}$.

29–40 □ Find the domain and sketch the graph of the function.

29. $f(x) = 3 - 2x$

30. $f(x) = x^2 + 2x - 1$

31. $g(x) = \sqrt{x - 5}$

32. $g(x) = \sqrt{6 - 2x}$

33. $G(x) = |x| + x$

34. $H(x) = |2x|$

35. $f(x) = x/|x|$

36. $f(x) = \dfrac{x^2 + 5x + 6}{x + 2}$

37. $f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x + 1 & \text{if } x > 0 \end{cases}$

38. $f(x) = \begin{cases} 2x + 3 & \text{if } x < -1 \\ 3 - x & \text{if } x \geq -1 \end{cases}$

39. $f(x) = \begin{cases} x + 2 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$

40. $f(x) = \begin{cases} -1 & \text{if } x \leq -1 \\ 3x + 2 & \text{if } |x| < 1 \\ 7 - 2x & \text{if } x \geq 1 \end{cases}$

41–46 □ Find an expression for the function whose graph is th given curve.

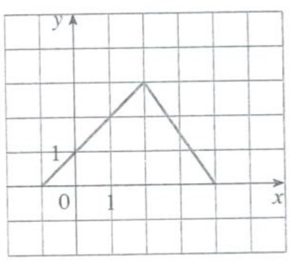
41. The line segment joining the points $(-2, 1)$ and $(4, -6)$

42. The line segment joining the points $(-3, -2)$ and $(6, 3)$

43. The bottom half of the parabola $x + (y - 1)^2 = 0$

44. The top half of the circle $(x - 1)^2 + y^2 = 1$

45.



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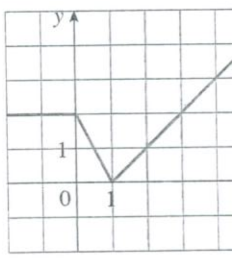


47–51 □ Find a formula for the described function and state its domain.

47. A rectangle has perimeter 20 m. Express the area of the rectangle as a function of the length of one of its sides.

48. A rectangle has area $16\ \text{m}^2$. Express the perimeter of the rectangle as a function of the length of one of its sides.

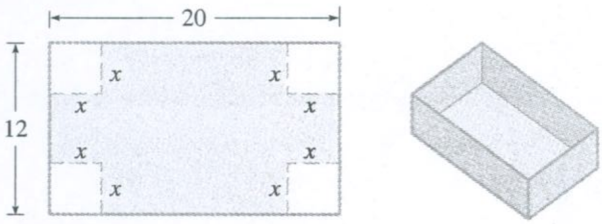
49. Express the area of an equilateral triangle as a function of the length of a side.

50. Express the surface area of a cube as a function of its volume.

51. An open rectangular box with volume $2\ \text{m}^3$ has a square base. Express the surface area of the box as a function of the length of a side of the base.

52. A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 30 ft, express the area A of the window as a function of the width x of the window.

53. A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 12 in. by 20 in. by cutting out equal squares of side x at each corner and then folding up the sides as in the figure. Express the volume V of the box as a function of x.



54. A taxi company charges two dollars for the first mile (or part of a mile) and 20 cents for each succeeding tenth of a mile (or part). Express the cost C (in dollars) of a ride as a function of the distance x traveled (in miles) for $0 < x < 2$, and sketch the graph of this function.

55. In a certain country, income tax is assessed as follows. There is no tax on income up to \$10,000. Any income over \$10,000 is taxed at a rate of 10%, up to an income of \$20,000. Any income over \$20,000 is taxed at 15%.

(a) Sketch the graph of the tax rate R as a function of the income I.

(b) How much tax is assessed on an income of \$14,000? On \$26,000?

(c) Sketch the graph of the total assessed tax T as a function of the income I.

56. The functions in Example 10 and Exercises 54 and 55(a) are called *step functions* because their graphs look like stairs. Give two other examples of step functions that arise in everyday life.

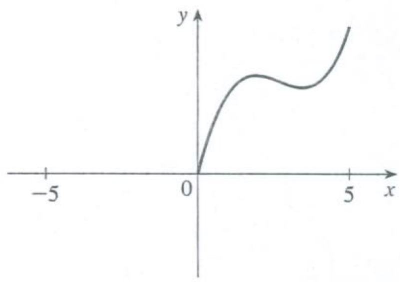
57. (a) If the point $(5, 3)$ is on the graph of an even function, what other point must also be on the graph?

(b) If the point $(5, 3)$ is on the graph of an odd function, what other point must also be on the graph?

58. A function f has domain $[-5, 5]$ and a portion of its graph is shown.

(a) Complete the graph of f if it is known that f is even.

(b) Complete the graph of f if it is known that f is odd.



59–64 Determine whether f is even, odd, or neither. If f is even or odd, use symmetry to sketch its graph.

59. $f(x) = x^{-2}$

60. $f(x) = x^{-3}$

61. $f(x) = x^2 + x$

62. $f(x) = x^4 - 4x^2$

63. $f(x) = x^3 - x$

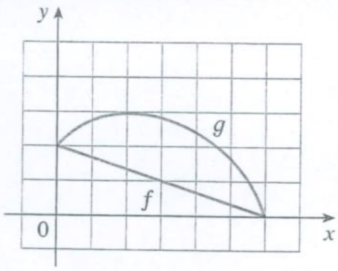
64. $f(x) = 3x^3 + 2x^2 + 1$

1.2 Mathematical Models

A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon such as the size of a population, the demand for a product, the speed of a falling object, the concentration of a product in a chemical reac-

29–30 □ Use graphical addition to sketch the graph of $f + g$.

29.



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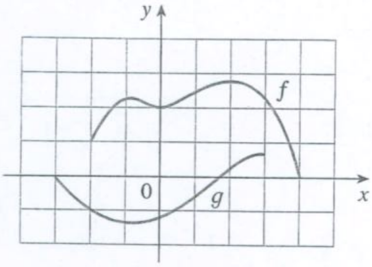


31–32 □ Find $f + g$, $f - g$, fg, and f/g and state their domains.

31. $f(x) = x^3 + 2x^2$, $g(x) = 3x^2 - 1$

32. $f(x) = \sqrt{1 + x}$, $g(x) = \sqrt{1 - x}$

33–34 □ Use the graphs of f and g and the method of graphical addition to sketch the graph of $f + g$.

33. $f(x) = x$, $g(x) = 1/x$ **34.** $f(x) = x^3$, $g(x) = -x^2$

35–40 □ Find the functions $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$ and their domains.

35. $f(x) = 2x^2 - x$, $g(x) = 3x + 2$

36. $f(x) = \sqrt{x - 1}$, $g(x) = x^2$

37. $f(x) = 1/x$, $g(x) = x^3 + 2x$

38. $f(x) = \dfrac{1}{x - 1}$, $g(x) = \dfrac{x - 1}{x + 1}$

39. $f(x) = \sin x$, $g(x) = 1 - \sqrt{x}$

40. $f(x) = \sqrt{x^2 - 1}$, $g(x) = \sqrt{1 - x}$

41–44 □ Find $f \circ g \circ h$.

41. $f(x) = x - 1$, $g(x) = \sqrt{x}$, $h(x) = x - 1$

42. $f(x) = \dfrac{1}{x}$, $g(x) = x^3$, $h(x) = x^2 + 2$

43. $f(x) = x^4 + 1$, $g(x) = x - 5$, $h(x) = \sqrt{x}$

44. $f(x) = \sqrt{x}$, $g(x) = \dfrac{x}{x - 1}$, $h(x) = \sqrt[3]{x}$

45–50 □ Express the function in the form $f \circ g$.

45. $F(x) = (x - 9)^5$ **46.** $F(x) = \sin(\sqrt{x})$

47. $G(x) = \dfrac{x^2}{x^2 + 4}$ **48.** $G(x) = \dfrac{1}{x + 3}$

49. $u(t) = \sqrt{\cos t}$ **50.** $u(t) = \tan \pi t$

51–53 □ Express the function in the form $f \circ g \circ h$.

51. $H(x) = 1 - 3^{x^2}$ **52.** $H(x) = \sqrt[3]{\sqrt{x} - 1}$

53. $H(x) = \sec^4(\sqrt{x})$

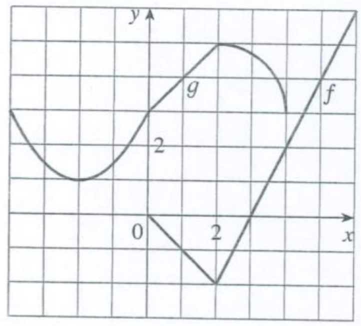
54. Use the table to evaluate each expression.

(a) $f(g(1))$ (b) $g(f(1))$ (c) $f(f(1))$
(d) $g(g(1))$ (e) $(g \circ f)(3)$ (f) $(f \circ g)(6)$

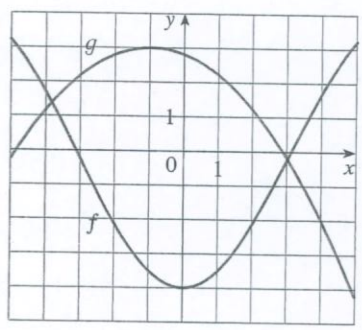
x	1	2	3	4	5	6
$f(x)$	3	1	4	2	2	5
$g(x)$	6	3	2	1	2	3

55. Use the given graphs of f and g to evaluate each expression, or explain why it is undefined.

(a) $f(g(2))$ (b) $g(f(0))$ (c) $(f \circ g)(0)$
(d) $(g \circ f)(6)$ (e) $(g \circ g)(-2)$ (f) $(f \circ f)(4)$



56. Use the given graphs of f and g to estimate the value of $f(g(x))$ for $x = -5, -4, -3, \ldots, 5$. Use these estimates to sketch a rough graph of $f \circ g$.



EXAMPLE 9 □ Find $\lim_{x \to 3^+} \frac{2}{x-3}$ and $\lim_{x \to 3^-} \frac{2}{x-3}$.

SOLUTION If x is close to 3 but larger than 3, then the denominator $x - 3$ is a small positive number and so $2/(x-3)$ is a large positive number. Thus, intuitively we see that

$$\lim_{x \to 3^+} \frac{2}{x-3} = \infty$$

Likewise, if x is close to 3 but smaller than 3, then $x - 3$ is a small negative number and so $2/(x-3)$ is a numerically large negative number. Thus

$$\lim_{x \to 3^-} \frac{2}{x-3} = -\infty$$

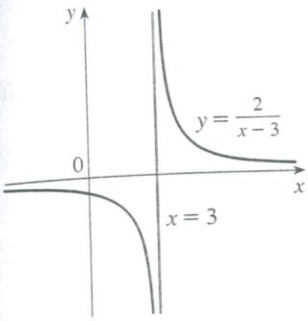


FIGURE 15

The graph of the curve $y = 2/(x-3)$ is given in Figure 15. The line $x = 3$ is a vertical asymptote.

EXAMPLE 10 □ Find the vertical asymptotes of $f(x) = \tan x$.

SOLUTION Because

$$\tan x = \frac{\sin x}{\cos x}$$

there are potential vertical asymptotes where $\cos x = 0$. In fact, since $\cos x \to 0^+$ as $x \to (\pi/2)^-$ and $\cos x \to 0^-$ as $x \to (\pi/2)^+$, whereas $\sin x$ is positive when x is near $\pi/2$, we have

$$\lim_{x \to (\pi/2)^-} \tan x = \infty \qquad \text{and} \qquad \lim_{x \to (\pi/2)^+} \tan x = -\infty$$

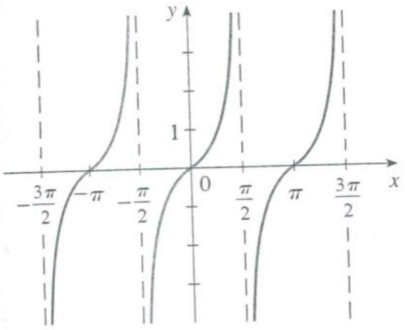


FIGURE 16
$y = \tan x$

This shows that the line $x = \pi/2$ is a vertical asymptote. Similar reasoning shows that the lines $x = (2n+1)\pi/2$, where n is an integer, are all vertical asymptotes of $f(x) = \tan x$. The graph in Figure 16 confirms this.

Another example of a function whose graph has a vertical asymptote is the natural logarithmic function $y = \ln x$. From Figure 17 we see that

$$\boxed{\lim_{x \to 0^+} \ln x = -\infty}$$

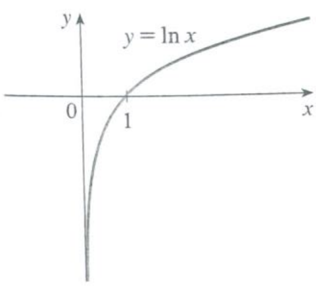


FIGURE 17
The y-axis is a vertical asymptote of the natural logarithmic function.

and so the line $x = 0$ (the y-axis) is a vertical asymptote. In fact, the same is true for $y = \log_a x$ provided that $a > 1$. (See Figures 11 and 12 in Section 1.6.)

2.2 Exercises

1. Explain in your own words what is meant by the equation

$$\lim_{x \to 2} f(x) = 5$$

Is it possible for this statement to be true and yet $f(2) = 3$? Explain.

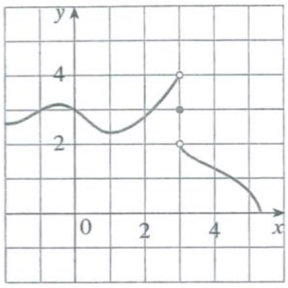
2. Explain what it means to say that

$$\lim_{x \to 1^-} f(x) = 3 \qquad \text{and} \qquad \lim_{x \to 1^+} f(x) = 7$$

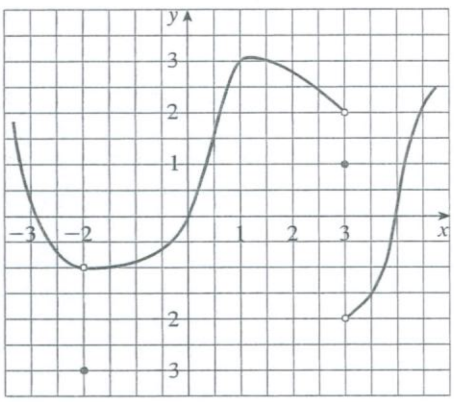
In this situation is it possible that $\lim_{x \to 1} f(x)$ exists? Explain.

3. Explain the meaning of each of the following.
(a) $\lim_{x\to-3} f(x) = \infty$ (b) $\lim_{x\to4^+} f(x) = -\infty$

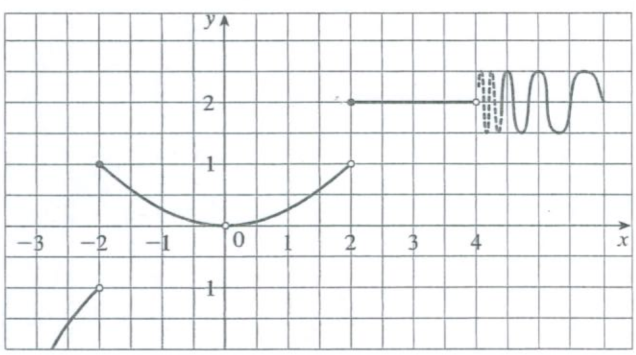
4. For the function f whose graph is given, state the value of the given quantity, if it exists. If it does not exist, explain why.
(a) $\lim_{x\to0} f(x)$ (b) $\lim_{x\to3^-} f(x)$
(c) $\lim_{x\to3^+} f(x)$ (d) $\lim_{x\to3} f(x)$
(e) $f(3)$



5. For the function f whose graph is given, state the value of the given quantity, if it exists. If it does not exist, explain why.
(a) $\lim_{x\to1} f(x)$ (b) $\lim_{x\to3^-} f(x)$ (c) $\lim_{x\to3^+} f(x)$
(d) $\lim_{x\to3} f(x)$ (e) $f(3)$ (f) $\lim_{x\to-2^-} f(x)$
(g) $\lim_{x\to-2^+} f(x)$ (h) $\lim_{x\to-2} f(x)$ (i) $f(-2)$

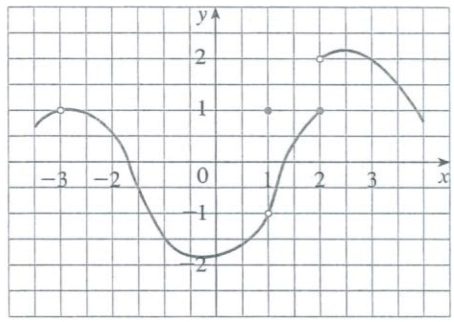


6. For the function g whose graph is given, state the value of the given quantity, if it exists. If it does not exist, explain why.
(a) $\lim_{x\to-2^-} g(x)$ (b) $\lim_{x\to-2^+} g(x)$ (c) $\lim_{x\to-2} g(x)$

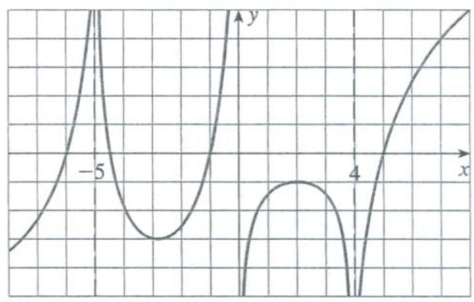


(d) $g(-2)$ (e) $\lim_{x\to2^-} g(x)$ (f) $\lim_{x\to2^+} g(x)$
(g) $\lim_{x\to2} g(x)$ (h) $g(2)$ (i) $\lim_{x\to4^+} g(x)$
(j) $\lim_{x\to4^-} g(x)$ (k) $g(0)$ (l) $\lim_{x\to0} g(x)$

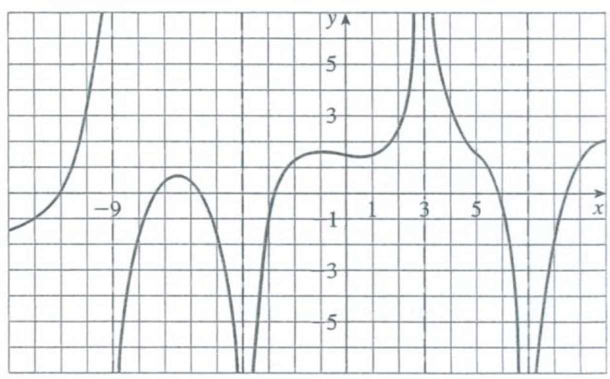
7. State the value of the limit, if it exists, from the given graph. If it does not exist, explain why.
(a) $\lim_{x\to3} f(x)$ (b) $\lim_{x\to1} f(x)$ (c) $\lim_{x\to-3} f(x)$
(d) $\lim_{x\to2^-} f(x)$ (e) $\lim_{x\to2^+} f(x)$ (f) $\lim_{x\to2} f(x)$



8. For the function g whose graph is shown, state the following.
(a) $\lim_{x\to-6} g(x)$ (b) $\lim_{x\to0^-} g(x)$
(c) $\lim_{x\to0^+} g(x)$ (d) $\lim_{x\to4} g(x)$
(e) The equations of the vertical asymptotes.



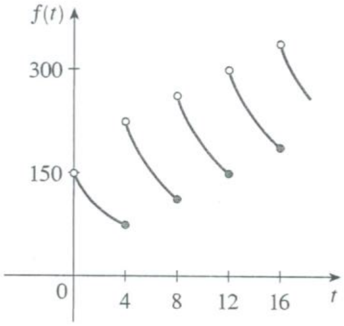
9. For the function f whose graph is shown, state the following.
(a) $\lim_{x\to3} f(x)$ (b) $\lim_{x\to7} f(x)$ (c) $\lim_{x\to-4} f(x)$
(d) $\lim_{x\to-9^-} f(x)$ (e) $\lim_{x\to-9^+} f(x)$
(f) The equations of the vertical asymptotes



10. A patient receives a 150-mg injection of a drug every 4 hours. The graph shows the amount $f(t)$ of the drug in the bloodstream after t hours. (Later we will be able to compute the dosage and time interval to ensure that the concentration of the drug does not reach a harmful level.) Find

$$\lim_{t \to 12^-} f(t) \qquad \text{and} \qquad \lim_{t \to 12^+} f(t)$$

and explain the significance of these one-sided limits.



11. Use the graph of the function $f(x) = 1/(1 + e^{1/x})$ to state the value of each limit, if it exists. If it does not exist, explain why.
(a) $\lim_{x \to 0^-} f(x)$ (b) $\lim_{x \to 0^+} f(x)$ (c) $\lim_{x \to 0} f(x)$

12. Sketch the graph of the following function and use it to determine the values of a for which $\lim_{x \to a} f(x)$ exists:

$$f(x) = \begin{cases} 2 - x & \text{if } x < -1 \\ x & \text{if } -1 \leqslant x < 1 \\ (x-1)^2 & \text{if } x \geqslant 1 \end{cases}$$

13–14 □ Sketch the graph of an example of a function f that satisfies all of the given conditions.

13. $\lim_{x \to 3^+} f(x) = 4$, $\lim_{x \to 3^-} f(x) = 2$, $\lim_{x \to -2} f(x) = 2$,
$f(3) = 3$, $f(-2) = 1$

14. $\lim_{x \to 0^-} f(x) = 1$, $\lim_{x \to 0^+} f(x) = -1$, $\lim_{x \to 2^-} f(x) = 0$
$\lim_{x \to 2^+} f(x) = 1$, $f(2) = 1$, $f(0)$ is undefined

15–20 □ Evaluate the function at the given numbers (correct to six decimal places). Use the results to guess the value of the limit, or explain why it does not exist.

15. $g(x) = \dfrac{x-1}{x^3-1}$;
$x = 0.2, 0.4, 0.6, 0.8, 0.9, 0.99, 1.8, 1.6, 1.4, 1.2, 1.1, 1.01$;
$\lim_{x \to 1} \dfrac{x-1}{x^3-1}$

16. $g(x) = \dfrac{1-x^2}{x^2+3x-10}$;
$x = 3, 2.1, 2.01, 2.001, 2.0001, 2.00001$;
$\lim_{x \to 2^+} \dfrac{1-x^2}{x^2+3x-10}$

17. $F(x) = \dfrac{(1/\sqrt{x}) - \frac{1}{5}}{x - 25}$;
$x = 26, 25.5, 25.1, 25.05, 25.01, 24, 24.5, 24.9, 24.95, 24.99$;
$\lim_{x \to 25} \dfrac{(1/\sqrt{x}) - \frac{1}{5}}{x - 25}$

18. $F(t) = \dfrac{\sqrt[3]{t} - 1}{\sqrt{t} - 1}$; $t = 1.5, 1.2, 1.1, 1.01, 1.001$; $\lim_{t \to 1} \dfrac{\sqrt[3]{t} - 1}{\sqrt{t} - 1}$

19. $f(x) = \dfrac{1 - \cos x}{x^2}$; $x = 1, 0.5, 0.4, 0.3, 0.2, 0.1, 0.05, 0.01$;
$\lim_{x \to 0} \dfrac{1 - \cos x}{x^2}$

20. $g(x) = \sqrt{x} \ln x$;
$x = 1, 0.5, 0.1, 0.05, 0.01, 0.005, 0.001$;
$\lim_{x \to 0^+} \sqrt{x} \ln x$

21–28 □ Determine the infinite limit.

21. $\lim_{x \to 5^+} \dfrac{6}{x-5}$

22. $\lim_{x \to 5^-} \dfrac{6}{x-5}$

23. $\lim_{x \to 3} \dfrac{1}{(x-3)^8}$

24. $\lim_{x \to 0} \dfrac{x-1}{x^2(x+2)}$

25. $\lim_{x \to -2^+} \dfrac{x-1}{x^2(x+2)}$

26. $\lim_{x \to \pi^-} \csc x$

27. $\lim_{x \to (-\pi/2)^-} \sec x$

28. $\lim_{x \to 5^+} \ln(x-5)$

29. Determine $\lim_{x \to 1^-} \dfrac{1}{x^3-1}$ and $\lim_{x \to 1^+} \dfrac{1}{x^3-1}$
(a) by evaluating $f(x) = 1/(x^3 - 1)$ for values of x that approach 1 from the left and from the right,
(b) by reasoning as in Example 9, and
(c) from a graph of f.

30. (a) Find the vertical asymptotes of the function

$$y = \frac{x}{x^2 - x - 2}$$

(b) Confirm your answer to part (a) by graphing the function.

31. (a) Estimate the value of the limit $\lim_{x \to 0} (1 + x)^{1/x}$ to five decimal places. Does this number look familiar?
(b) Illustrate part (a) by graphing the function $y = (1 + x)^{1/x}$.

32. The slope of the tangent line to the graph of the exponential function $y = 2^x$ at the point $(0, 1)$ is $\lim_{x \to 0} (2^x - 1)/x$. Estimate the slope to three decimal places.

33. (a) By graphing the function $f(x) = (\tan 4x)/x$ and zooming in toward the point where the graph crosses the y-axis, estimate the value of $\lim_{x \to 0} f(x)$.

EXAMPLE 11 □ Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$.

SOLUTION First note that we *cannot* use

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = \lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \sin \frac{1}{x}$$

because $\lim_{x \to 0} \sin(1/x)$ does not exist (see Example 4 in Section 2.2). However, since

$$-1 \leqslant \sin \frac{1}{x} \leqslant 1$$

we have, as illustrated by Figure 8,

$$-x^2 \leqslant x^2 \sin \frac{1}{x} \leqslant x^2$$

Watch an animation of a similar limit.
Resources / Module 2
/ Basics of Limits
/ Sound of a Limit that Exists

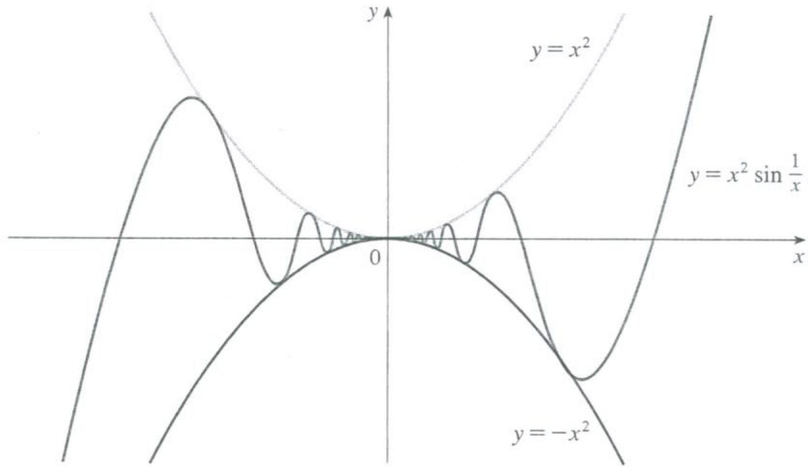


FIGURE 8

We know that

$$\lim_{x \to 0} x^2 = 0 \qquad \text{and} \qquad \lim_{x \to 0} -x^2 = 0$$

Taking $f(x) = -x^2$, $g(x) = x^2 \sin(1/x)$, and $h(x) = x^2$ in the Squeeze Theorem, we obtain

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$$

□

2.3 Exercises

1. Given that

$$\lim_{x \to a} f(x) = -3 \qquad \lim_{x \to a} g(x) = 0 \qquad \lim_{x \to a} h(x) = 8$$

find the limits that exist. If the limit does not exist, explain why.

(a) $\lim_{x \to a} [f(x) + h(x)]$

(b) $\lim_{x \to a} [f(x)]^2$

(c) $\lim_{x \to a} \sqrt[3]{h(x)}$

(d) $\lim_{x \to a} \frac{1}{f(x)}$

(e) $\lim_{x \to a} \frac{f(x)}{h(x)}$

(f) $\lim_{x \to a} \frac{g(x)}{f(x)}$

(g) $\lim_{x \to a} \frac{f(x)}{g(x)}$

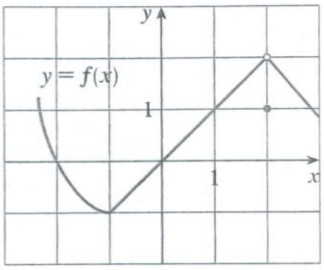
(h) $\lim_{x \to a} \frac{2f(x)}{h(x) - f(x)}$

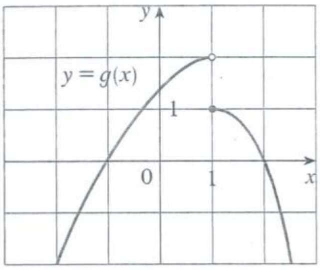
2. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.

(a) $\lim_{x\to 2}[f(x)+g(x)]$ (b) $\lim_{x\to 1}[f(x)+g(x)]$

(c) $\lim_{x\to 0}[f(x)g(x)]$ (d) $\lim_{x\to -1}\dfrac{f(x)}{g(x)}$

(e) $\lim_{x\to 2}x^3f(x)$ (f) $\lim_{x\to 1}\sqrt{3+f(x)}$





3–9 □ Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

3. $\lim_{x\to 4}(5x^2-2x+3)$

4. $\lim_{x\to 3}(x^3+2)(x^2-5x)$

5. $\lim_{x\to -1}\dfrac{x-2}{x^2+4x-3}$

6. $\lim_{x\to 1}\left(\dfrac{x^4+x^2-6}{x^4+2x+3}\right)^2$

7. $\lim_{t\to -2}(t+1)^9(t^2-1)$

8. $\lim_{u\to -2}\sqrt{u^4+3u+6}$

9. $\lim_{x\to 4^-}\sqrt{16-x^2}$

10. (a) What is wrong with the following equation?

$$\frac{x^2+x-6}{x-2}=x+3$$

(b) In view of part (a), explain why the equation

$$\lim_{x\to 2}\frac{x^2+x-6}{x-2}=\lim_{x\to 2}(x+3)$$

is correct.

11–28 □ Evaluate the limit, if it exists.

Resources / Module 2 / Basics of Limits / Problems and Tests

11. $\lim_{x\to -3}\dfrac{x^2-x+12}{x+3}$

12. $\lim_{x\to -3}\dfrac{x^2-x-12}{x+3}$

13. $\lim_{x\to -2}\dfrac{x+2}{x^2-x-6}$

14. $\lim_{x\to 1}\dfrac{x^2+x-2}{x^2-3x+2}$

15. $\lim_{h\to 0}\dfrac{(h-5)^2-25}{h}$

16. $\lim_{x\to 1}\dfrac{x^3-1}{x^2-1}$

17. $\lim_{h\to 0}\dfrac{(1+h)^4-1}{h}$

18. $\lim_{h\to 0}\dfrac{(2+h)^3-8}{h}$

19. $\lim_{t\to 9}\dfrac{9-t}{3-\sqrt{t}}$

20. $\lim_{t\to 2}\dfrac{t^2+t-6}{t^2-4}$

21. $\lim_{t\to 0}\dfrac{\sqrt{2-t}-\sqrt{2}}{t}$

22. $\lim_{x\to 2}\dfrac{x^4-16}{x-2}$

23. $\lim_{x\to 9}\dfrac{x^2-81}{\sqrt{x}-3}$

24. $\lim_{x\to 1}\left[\dfrac{1}{x-1}-\dfrac{2}{x^2-1}\right]$

25. $\lim_{t\to 0}\left[\dfrac{1}{t\sqrt{1+t}}-\dfrac{1}{t}\right]$

26. $\lim_{h\to 0}\dfrac{(3+h)^{-1}-3^{-1}}{h}$

27. $\lim_{x\to 2}\dfrac{\dfrac{1}{x}-\dfrac{1}{2}}{x-2}$

28. $\lim_{x\to 1}\dfrac{\sqrt{x}-x^2}{1-\sqrt{x}}$

29. (a) Estimate the value of

$$\lim_{x\to 0}\frac{x}{\sqrt{1+3x}-1}$$

by graphing the function $f(x)=x/(\sqrt{1+3x}-1)$.

(b) Make a table of values of $f(x)$ for x close to 0 and guess the value of the limit.

(c) Use the Limit Laws to prove that your guess is correct.

30. (a) Use a graph of

$$f(x)=\frac{\sqrt{3+x}-\sqrt{3}}{x}$$

to estimate the value of $\lim_{x\to 0}f(x)$ to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Use the Limit Laws to find the exact value of the limit.

31. Use the Squeeze Theorem to show that $\lim_{x\to 0}x^2\cos 20\pi x=0$. Illustrate by graphing the functions $f(x)=-x^2$, $g(x)=x^2\cos 20\pi x$, and $h(x)=x^2$ on the same screen.

32. Use the Squeeze Theorem to show that

$$\lim_{x\to 0}\sqrt{x^3+x^2}\sin\frac{\pi}{x}=0$$

Illustrate by graphing the functions f, g, and h (in the notation of the Squeeze Theorem) on the same screen.

33. If $1\leqslant f(x)\leqslant x^2+2x+2$ for all x, find $\lim_{x\to -1}f(x)$.

34. If $3x\leqslant f(x)\leqslant x^3+2$ for $0\leqslant x\leqslant 2$, evaluate $\lim_{x\to 1}f(x)$.

35. Prove that $\lim_{x\to 0}x^4\cos\dfrac{2}{x}=0$.

36. Prove that $\lim_{x\to 0^+}\sqrt{x}\,e^{\sin(\pi/x)}=0$.

37–42 □ Find the limit, if it exists. If the limit does not exist, explain why.

37. $\lim_{x\to -4}|x+4|$

38. $\lim_{x\to -4^-}\dfrac{|x+4|}{x+4}$

39. $\lim_{x\to 2}\dfrac{|x-2|}{x-2}$

40. $\lim_{x\to 1.5}\dfrac{2x^2-3x}{|2x-3|}$

41. $\lim_{x \to 0^-} \left(\frac{1}{x} - \frac{1}{|x|} \right)$

42. $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{|x|} \right)$

43. The *signum* (or sign) *function*, denoted by sgn, is defined by

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

(a) Sketch the graph of this function.
(b) Find each of the following limits or explain why it does not exist.
(i) $\lim_{x \to 0^+} \operatorname{sgn} x$ (ii) $\lim_{x \to 0^-} \operatorname{sgn} x$
(iii) $\lim_{x \to 0} \operatorname{sgn} x$ (iv) $\lim_{x \to 0} |\operatorname{sgn} x|$

44. Let

$$f(x) = \begin{cases} x^2 - 2x + 2 & \text{if } x < 1 \\ 3 - x & \text{if } x \geqslant 1 \end{cases}$$

(a) Find $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$.
(b) Does $\lim_{x \to 1} f(x)$ exist?
(c) Sketch the graph of f.

45. Let $F(x) = \frac{x^2 - 1}{|x - 1|}$.
(a) Find
(i) $\lim_{x \to 1^+} F(x)$ (ii) $\lim_{x \to 1^-} F(x)$
(b) Does $\lim_{x \to 1} F(x)$ exist?
(c) Sketch the graph of F.

46. Let

$$h(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } 0 < x \leqslant 2 \\ 8 - x & \text{if } x > 2 \end{cases}$$

(a) Evaluate each of the following limits, if it exists.
(i) $\lim_{x \to 0^+} h(x)$ (ii) $\lim_{x \to 0} h(x)$ (iii) $\lim_{x \to 1} h(x)$
(iv) $\lim_{x \to 2^-} h(x)$ (v) $\lim_{x \to 2^+} h(x)$ (vi) $\lim_{x \to 2} h(x)$
(b) Sketch the graph of h.

47. (a) If the symbol $[\![\ ]\!]$ denotes the greatest integer function defined in Example 10, evaluate
(i) $\lim_{x \to -2^+} [\![x]\!]$ (ii) $\lim_{x \to -2} [\![x]\!]$ (iii) $\lim_{x \to -2.4} [\![x]\!]$
(b) If n is an integer, evaluate
(i) $\lim_{x \to n^-} [\![x]\!]$ (ii) $\lim_{x \to n^+} [\![x]\!]$
(c) For what values of a does $\lim_{x \to a} [\![x]\!]$ exist?

48. Let $f(x) = x - [\![x]\!]$.
(a) Sketch the graph of f.
(b) If n is an integer, evaluate
(i) $\lim_{x \to n^-} f(x)$ (ii) $\lim_{x \to n^+} f(x)$
(c) For what values of a does $\lim_{x \to a} f(x)$ exist?

49. If $f(x) = [\![x]\!] + [\![-x]\!]$, show that $\lim_{x \to 2} f(x)$ exists but is not equal to $f(2)$.

50. In the theory of relativity, the Lorentz contraction formula

$$L = L_0 \sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v \to c^-} L$ and interpret the result. Why is a left-hand limit necessary?

51. If p is a polynomial, show that $\lim_{x \to a} p(x) = p(a)$.

52. If r is a rational function, use Exercise 51 to show that $\lim_{x \to a} r(x) = r(a)$ for every number a in the domain of r.

53. If

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x \to 0} f(x) = 0$.

54. Show by means of an example that $\lim_{x \to a} [f(x) + g(x)]$ may exist even though neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exists.

55. Show by means of an example that $\lim_{x \to a} [f(x)g(x)]$ may exist even though neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exists.

56. Evaluate $\lim_{x \to 2} \frac{\sqrt{6 - x} - 2}{\sqrt{3 - x} - 1}$.

57. Is there a number a such that

$$\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of a and the value of the limit.

58. The figure shows a fixed circle C_1 with equation $(x - 1)^2 + y^2 = 1$ and a shrinking circle C_2 with radius r and center the origin. P is the point $(0, r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_2 shrinks, that is, as $r \to 0^+$?