Last time: infinite limits
limit laws

Today: Cancellation
Squeeze Theorem

Cancellation:

EX] $\lim_{x \to -3} \frac{x^2 - x - 12}{x + 3} = \lim_{x \to -3} \frac{(x-4)\cancel{(x+3)}}{\cancel{x+3}}$

$= \lim_{x \to -3} x - 4$

$= -7.$ ✓

(why? We don't care about $x = -3$ when computing $\lim_{x \to -3}$)

EX] Compute $\lim_{h \to 0} \frac{(3+h)^{-1} - 3^{-1}}{h}$.

So: $\lim_{h \to 0} \frac{(3+h)^{-1} - 3^{-1}}{h} = \lim_{h \to 0} \frac{\frac{1}{(3+h)} - \frac{1}{3}}{h}$

common denominator is $3(3+h)$

$3 - 3 - h$

$= \lim_{h \to 0} \frac{\frac{3}{3(3+h)} - \frac{(3+h)}{3(3+h)}}{h}$

$= \lim_{h \to 0} \frac{\frac{-h}{3(3+h)}}{h}$

Recall: $\left(\frac{a}{b} = a \cdot \frac{1}{b} \right)$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{3(3+h)} \cdot \frac{1}{\cancel{h}}$$

$$= \lim_{h \to 0} \frac{-1}{3(3+h)} = -\frac{1}{9}$$

↳ use Limit Laws.

Before the next example, recall the absolute value function:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

EX] Compute $\lim\limits_{x \to 3} \frac{|x-3|}{x-3}$.

Recall: $\lim\limits_{x \to a} f(x) = L \iff \lim\limits_{x \to a^-} f(x) = L$ and

$\lim\limits_{x \to a^+} f(x) = L$.

Check from the left and right! Note that

$$|x-3| = \begin{cases} x-3 & \text{if } x-3 \geq 0 \iff x \geq 3 \leftarrow \\ -(x-3) & \text{if } x-3 < 0 \leftarrow \\ & \iff x < 3. \end{cases}$$

Left: $\lim\limits_{x \to 3^-} \frac{|x-3|}{x-3} = \lim\limits_{x \to 3^-} \frac{-\cancel{(x-3)}}{\cancel{x-3}}$

↳ means $x < 3$

$$= \lim_{x \to 3^-} -1 = -1.$$

Right: $\lim\limits_{x\to 3^+} \frac{|x-3|}{x-3} = \lim\limits_{x\to 3^+} \frac{\cancel{x-3}}{\cancel{x-3}} = \lim\limits_{x\to 3^+} 1$

↓ means $x > 3$

$= 1.$

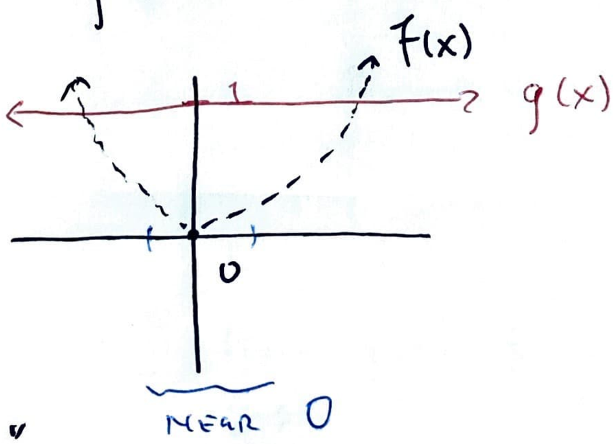
Since $\lim\limits_{x\to 3^-} \neq \lim\limits_{x\to 3^+} \Rightarrow \lim\limits_{x\to 3} \frac{|x-3|}{x-3}$ DNE.

Now: Squeeze Theorem.

First, some motivation.

Ex] Consider $f(x) = x^2$

$g(x) = 1$



Note: "around 0" $\Rightarrow$ $f(x) \leq g(x)$.

Now: $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} x^2 = 0 \leq 1 = \lim\limits_{x\to 0} 1$

$= \lim\limits_{x\to 0} g(x).$

In general ---

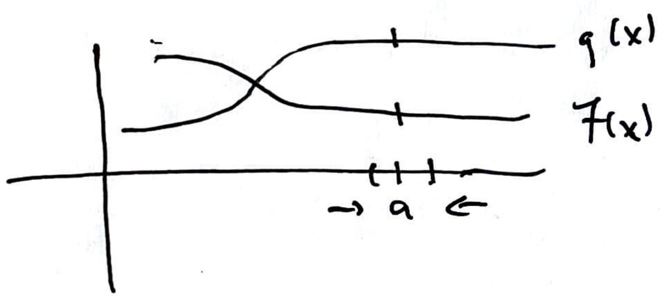
Theorem ("THM") Suppose we have

$$f(x) \le g(x) \quad \text{(near } a\text{)}$$

and $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ exist. Then

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

(Picture:



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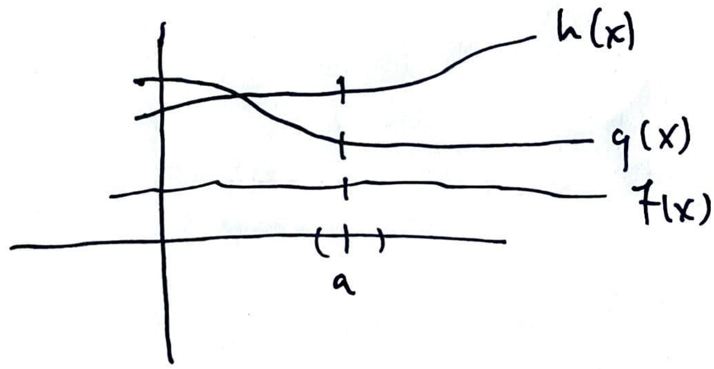
Now: What if



By the THM,

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x) \le \lim_{x\to a} h(x).$$

Squeeze: If $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} h(x) = L$

forces $\Rightarrow \lim_{x\to a} g(x) = L$.

THM ("Squeeze Thm") If $f(x) \le g(x) \le h(x)$ where x is near a and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$

then

$$\lim_{x\to a} g(x) = L.$$

EX] Prove that $\lim_{x\to 0} x^2 \sin x = 0$.

Recall that: $-1 \le \sin x \le 1$

Multiply by x^2 $\Rightarrow$ $-x^2 \le x^2 \sin x \le x^2$

(ok since $x^2 \ge 0$)

Now: since $\lim_{x\to 0} -x^2 = 0 = \lim_{x\to 0} x^2$, by the Squeeze Theorem, $\lim_{x\to 0} x^2 \sin x = 0$.

Continuity:

Let's go back.. recall..

EX] $\lim_{x\to 2} x^2+1 = (2)^2+1 = 5$.

Note: x^2+1 is an example of ...

Def. A polynomial $P(x)$ is a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where the coefficients $a_0, a_1, \dots, a_n$ are real numbers and the degree n $(n \geq 0)$.

Thus: Polynomials have this nice property that

$$\lim_{x \to a} P(x) = \underbrace{P(a)}_{\hookrightarrow \text{ sub } x = a}.$$

Questions: What other functions have this property?

Ans: "Continuous functions".